THE ANNUAL MEETING OF THE SOCIETY

The forty-sixth Annual Meeting of the American Mathematical Society was held at the Ohio State University, Columbus, Ohio, from Tuesday to Friday, December 26–29, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the Mathematical Association of America, and the National Council of Teachers of Mathematics. This was the fifth time the Society has met at the Ohio State University, the other meetings being in the years 1899, 1915, 1926, and 1931.

There were four general and nine sectional sessions of the Society, at which eight addresses and one hundred two research papers (sixty-two in person) were given. The Gibbs Lecture on Wednesday afternoon was given in Campbell Hall; the joint session on Friday morning with Section E was held in Mendenhall Hall; all other sessions were held in the Horticulture and Forestry Building of the Ohio State University.

Arrangements for the meetings were made by a committee consisting of Professors S. E. Rasor (Chairman), Lincoln LaPaz, J. H. Weaver, and F. B. Wiley. The dormitory and social rooms of Neil Hall were made available for the use of the visiting mathematicians.

Over four hundred persons registered including the following two hundred eighty-six members of the Society, the largest attendance at an annual meeting in the history of the Society:

V. W. Adkisson, R. P. Agnew, Leonidas Alaoglu, A. A. Albert, G. E. Albert, H. W. Alexander, C. B. Allendoerfer, W. E. Anderson, Emil Artin, Max Astrachan, J. V. Atanasoff, C. S. Atchison, H. T. R. Aude, Frank Ayres, W. L. Ayres, R. W. Babcock, F. R. Bamforth, G. M. Bareis, I. A. Barnett, R. C. F. Bartels, E. F. Beckenbach, E. R. Beckwith, M. M. Beenken, A. A. Bennett, S. F. Bibb, G. D. Birkhoff, H. L. Black, Archie Blake, Henry Blumberg, L. M. Blumenthal, R. P. Boas, Salomon Bochner, Paul Boeder, D. G. Bourgin, J. W. Bower, M. G. Boyce, J. W. Bradshaw, J. B. Brandeberry, A. T. Brauer, W. C. Brenke, F. L. Brooks, O. E. Brown, G. S. Bruton, C. T. Bumer, R. S. Burington, Jewell H. Bushey, W. E. Byrne, W. D. Cairns, J. W. Calkin, H. H. Campaigne, H. C. Carter, W. B. Carver, E. W. Chittenden, R. V. Churchill, E. H. Clarke, W. W. S. Claytor, R. F. Clippinger, L. M. Coffin, L. W. Cohen, J. B. Coleman, A. H. Copeland, W. A. Cordrey, A. P. Cowgill, L. C. Cox, M. M. Culver, H. B. Curtis, J. H. Curtiss, A. H. Diamond, R. P. Dilworth, L. L. Dines, H. A. DoBell, J. L. Doob, H. H. Downing, W. C. Doyle, Arnold Dresden, R. J. Duffin, P. S. Dwyer, J. J. Eachus, E. D. Eaves, Samuel Eilenberg, W. H. Erskine, G. C. Evans, G. W. Evans, H. P. Evans, F. A. Ficken, B. F. Finkel, F. A. Foraker, L. R. Ford, W. B. Ford, Orrin Frink, T. C. Fry, D. G. Fulton, A. S. Galbraith, M. C. Garvin, B. E. Gatewood, H. M. Gehman, Abe Gelbart, F. J, Gerst, Lachlan Gilchrist, Wallace Givens, L. M. Graves, J. W. Green, V. G. Grove.

D. W. Hall, N. A. Hall, P. R. Halmos, O. G. Harrold, M. C. Hartley, M. L. Hartung, G. G. Harvey, O. C. Hazlett, E. R. Hedrick, A. E. Heins, M. H. Heins, E. D. Hellinger, Fritz Herzog, M. R. Hestenes, H. C. Hicks, J. J. L. Hinrichsen, D. L. Holl, T. R. Hollcroft, P. M. Hummel, E. V. Huntington, W. A. Hurwitz, D. H. Hyers, Dunham Jackson, R. D. James, E. D. Jenkins, Walter Jennings, Fritz John, R. P. Johnson, F. E. Johnston, B. W. Jones, H. A. Jordan, Mark Kac, Samuel Kaplan, Wilfred Kaplan, L. C. Karpinski, Chosaburo Kato, D. K. Kazarinoff, J. L. Kelley, R. B. Kershner, J. R. Kline, L. A. Knowler, Alfred Korzybski, H. W. Kuhn, A. C. Ladner, W. D. Lambert, K. W. Lamson, O. E. Lancaster, H. G. Landau, R. E. Langer, H. L. Langhaar, Lincoln LaPaz, C. G. Latimer, C. E. Leach, Solomon Lefschetz, D. H. Lehmer, R. A. Leibler, J. H. Levin, Norman Levinson, G. H. Ling, Marie Litzinger, M. I. Logsdon, A. T. Lonseth, E. R. Lorch, C. I. Lubin, P. H. McGrath, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, M. M. Maloney, J. D. Mancill, Dorothy Manning, R. G. Mason, Karl Menger, A. N. Milgram, D. D. Miller, E. W. Miller, Virginia Modesitt, M. G. Moore, Max Morris, Richard Morris, R. C. Morrow, Marston Morse, E. J. Moulton, J. R. Musselman, F. C. Ogg, Rufus Oldenburger, E. G. Olds, E. J. Olson, E. R. Ott, J. R. Overman, W. V. Parker, E. W. Paxson, D. T. Perkins, O. J. Peterson, Everett Pitcher, V. C. Poor, G. B. Price, Tibor Radó, J. F. Randolph, S. E. Rasor, F. W. Reed, Eric Reissner, R. G. D. Richardson, C. E. Rickart, P. R. Rider, R. F. Rinehart, H. P. Robertson, W. H. Roever, J. B. Rosenbach, J. E. Rosenthal, Barkley Rosser, E. H. Rothe, N. E. Rutt, O. F. G. Schilling, I. J. Schoenberg, G. E. Schweigert, C. E. Sealander, Seymour Sherman, C. G. Shover, D. R. Shreve, D. T. Sigley, M. E. Sinclair, E. R. Sleight, M. M. Slotnick, M. F. Smiley, E. R. Smith, F. C. Smith, J. P. Smith, T. L. Smith, W. F. Smith, Virgil Snyder, D. C. Spencer, G. W. Starcher, N. E. Steenrod, H. E. Stelson, C. N. Stokes, R. W. Stokes, E. C. Stopher, W. T. Stratton, E. C. Strayhorn, Otto Szász, J. S. Taylor, M. E. Taylor, W. C. Taylor, H. P. Thielman, J. M. Thomas, R. M. Thrall, E. W. Titt, C. C. Torrance, H. C. Trimble, W. J. Trjitzinsky, P. L. Trump, A. W. Tucker, F. W. Urban, J. L. Vanderslice, J. H. Van Vleck, J. I. Vass, R. W. Wagner, G. W. Walker, A. D. Wallace, J. L. Walsh, S. E. Warschawski, J. H. Weaver, Warren Weaver, M. S. Webster, M. J. Weiss, C. P. Wells, M E. Wescott, G. T. Whyburn, Norbert Wiener, L. R. Wilcox, R. L. Wilder, F. B. Wiley, C. W. Williams, K. P. Williams, E. W. Wilson, Aurel Wintner, F. L. Wren, C. R. Wylie, M. M. Young, J. W. T. Youngs, R. T. Zoch, Max Zorn.

The meeting opened at 2:00 Tuesday afternoon with sections for Analysis and for Topology. Professors L. L. Dines and G. T. Whyburn presided at these two sessions.

Provost E. R. Hedrick presided at the general session on Wednesday morning. On Wednesday afternoon there were three sessions, for Analysis, for Algebra, and for Logic and the Theory of Sets. Professors M. R. Hestenes, C. C. MacDuffee, and L. M. Blumenthal presided. On Thursday afternoon there were sessions for Applications and for Topology, at which Professors Norbert Wiener and Solomon Lefschetz presided. On Friday morning Professor Arnold Dresden presided at the section for Analysis.

On Wednesday afternoon Professor Theodore von Kármán,

Director of the Guggenheim Aeronautical Laboratory of the California Institute of Technology, delivered the fifteenth Josiah Willard Gibbs Lecture before a joint session of the Society and the American Association for the Advancement of Science. His subject was *The engineer grappling with non-linear problems.* Professor G. C. Evans, President of the Society, presided at this lecture.

The opening session of the American Association for the Advancement of Science was held on Wednesday evening at which Dr. W. C. Mitchell delivered the retiring presidential address on *The public relations of science.* This was followed by a reception for all members of the A.A.A.S. and affiliated societies at the Deshler-Wallick Hotel.

The general session on Wednesday morning opened with the business meeting and annual election. At this time the Frank Nelson Cole Prize in Algebra was awarded to Professor A. A. Albert for his papers *On the construction of Riemann matrices*, published in the Annals of Mathematics, vol. 35 (1934), pp. 1–28; vol. 36 (1935), pp. 376–394. This was followed by a joint session of the Society, the Association and Section A of the A.A.A.S. at which Professor J. R. Kline, as retiring Vice President of the A.A.A.S. and Chairman of Section A, gave an address entitled *The Jordan curve theorem.* The President of the Society, Professor G. C. Evans, presided.

At the general session on Thursday afternoon, by invitation of the Committee on Invited Speakers, Professor D. H. Lehmer delivered an address on *The application of Bernoulli polynomials to some problems in Diophantine analysis.* Professor H. W. Kuhn presided at this lecture.

On Friday morning the section for Applied Mathematics met jointly with the Association and Sections A and E of the A.A.A.S. At this session the following five addresses were given: *The beginnings of mathematical geophysics in Great Britain* by W. D. Lambert, *Use of mathematics in the delineation of magnetic and electric anomalies* by Professor Lachlan Gilchrist, *Gravimetric and seismic methods in exploratory geophysics* by Dr. M. M. Slotnick, *Mathematical problems in seismology* by Dr. Archie Blake, *Some seismological problems* by Professor Perry Byerly. In the absence of Professor Byerly, his prepared address was read by Professor W. D. Cairns. Professor Marston Morse, Chairman of Section A, presided at this joint session.

Sessions of the Mathematical Association were held on Friday afternoon and Saturday morning, and sessions of the National Council of Teachers on Friday morning and afternoon.

A joint dinner for the mathematical organizations and their guests

was held at the Faculty Club on Thursday evening. The attendance was two hundred eighty-three. Professor Tibor Radó was toastmaster and introduced Dean B. L. Stradley who welcomed the visitors to Columbus and the Ohio State University. Professor H. C. Christofferson called attention to some interesting trends in enrollments of grade school, secondary and college students in the past fifty years. Professor R. E. Langer discussed some proposed policies of the Association. Professor G. D. Birkhoff spoke of the origin and workings of the Putnam Prize Competition. Professor J. M. Thomas pointed to the vigorous but alarming growth in published research in this country and proposed that authors "Write less better." At the close of the dinner Dean G. H. Ling presented resolutions, which were adopted unanimously, expressing the appreciation of the mathematical groups to the Ohio State University and the local committee for the excellent arrangements and generous hospitality.

The ladies of the local Department of Mathematics served tea in Neil Hall on Tuesday and Friday afternoons for visiting mathematicians and guests. There was also a tea at the Governor's Mansion on Thursday afternoon.

At the meeting of the Board of Trustees at 7:30 P.M., Tuesday, December 26, in a social room of Neil Hall, there was not a quorum present. An adjourned meeting, therefore, was held on January 1, at 12:30 P.M., in the Men's Faculty Club of Columbia University, New York City.

The Council met on Wednesday, December 27, at 7:30 P.M. in a social room of Neil Hall. An adjourned meeting was held on Thursday at 11:15 P.M.

The Secretary announced the election of the following twenty-eight persons to membership in the Society:

Sister Anastasia Maria, Immaculata College, Immaculata, Pa.;
Miss Lois E. Bell, Independence Junior College, Independence, Kan.;
Mr. Kenneth Arthur Bush, Bard College, Annandale-on-Hudson, N.Y.;
Miss Mary Louise Constable, Philadelphia High School for Girls;
Mr. Commodore Columbus Dearman, Jr., East Central Junior College, Decatur, Miss.;
Mr. Bernard Dimsdale, University of Idaho;
Professor Solomon Dobrin, Jr., Essex Junior College, Newark, N.J.;
Mr. William Holt Glenn, Jr., Pasadena Junior College;
Dr. Michael Golomb, School of Electrical Engineering, Cornell University;
Professor Alberto Gonzalez Dominguez, Brown University;
Dr. Albert Edward Heins, Purdue University;
Dr. Stephen Arthur Jennings, Yale University;
Professor Pete Singleton Leach, Middle Georgia College, Cochran, Ga.;
Mr. Joseph Harmon Levin, Wayne University;

Mr. Hunter John McConnell, Salt Lake City, Utah;
Reverend Philip H. McGrath, St. Peter's College, Jersey City, N.J.;
Mr. Thomas Henry Murtaugh, Fordham University;
Dr. Andrewa R. Noble, Berkeley, Calif.;
Dr. Eric Reissner, Massachusetts Institute of Technology;
Mr. W. Murray Robertson, Pennsylvania Railroad, Narberth, Pa.;
Dr. Louis D. Rodabaugh, University of Alabama;
Dr. Peter Scherk, Taft School, Watertown, Conn.;
Mr. Morris Herbert Shamos, New York, N.Y.;
Professor Herman Walton Smith, Oklahoma Agricultural and Mechanical College;
Dr. Chester Snow, National Bureau of Standards, Washington, D.C.;
Dr. Donald Clayton Spencer, Massachusetts Institute of Technology;
Dr. Charles P. Wells, Michigan State College;
Mr. Max Wyman, California Institute of Technology.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Bowdoin College: Mr. Dan Edwin Christie (at Princeton University).
Brooklyn College: Mrs. Mary I. Draper Boeker.
Brown University: Dr. Willy K. Feller, Messrs. George Elmer Forsythe, John George Herriot, Joseph Albert Joseph, and Olaf Schmidt.
Bryn Mawr College: Dr. Hilda Geiringer and Miss Dorothy Maharam.
University of California: Mr. Douglas George Chapman and Miss Orla Virginia Wood.
University of California at Los Angeles: Messrs. Jule Gregory Charney and Harry Lass.
University of Chicago: Miss Katherine Elizabeth Hazard, Messrs. William Karush and Herman Lewis Meyer, Jr.
College of the City of New York: Mr. Kenneth J. Arrow.
Columbia University: Messrs. Seymour Jablon, Robert Alex McKean, Don Mittleman, and Fred Supnick.
Cornell University: Professor Walter Rue Murray (Franklin and Marshall College) and Mr. Edward Oscar Stephany.
Harvard University: Messrs. Maurice Haskell Heins and Edwin Norman Nilson, and Hon. John Lord O'Brien (of Buffalo, N.Y.).
University of Illinois: Messrs. David Harold Blackwell, Abraham Charnes, Robert W. Gibson, Nathan Goldman, and Yudell Leo Luke.
Indiana University: Mr. David Gilbarg and Miss Margaret Ellen Stump.
Institute for Advanced Study: Drs. Alfred Theodor Brauer and Clifford Hugh Dowker, Professor Guido Fubini, Mr. Albert Edward Ingham (King's College, Cambridge, England), and Professor Charles Eugene Springer (University of Oklahoma).
The State University of Iowa: Mr. Carl Elroy Noble.
Iowa State College: Mr. Roy Herbert Cook.
The Johns Hopkins University: Mr. Samuel G. Bourne, Miss Florence Jessie Collinson, Mr. Alfred Levine, Miss Yael Naim, and Mr. Harold Kenneth Sohl.
University of Kentucky: Mr. William Glenn Clark.
Lehigh University: Mr. Eugene Park.
Massachusetts Institute of Technology: Messrs. Francis Begnaud Hildebrand, Carl Gustaf Allan Nordling, and Claude E. Shannon.

University of Michigan: Dr. Samuel Eilenberg and Mr. Sheng Chin Fan.
University of Minnesota: Miss Margaret Pearl Martin, Messrs. William Robert McEwen and Frank Joseph Polansky.
Northwestern University: Messrs. Theodore Wilbur Anderson, Jr., Harold Elra Burns (Indiana University Extension, East Chicago, Ind.), Thomas Elias Caywood, Edwin Leonidas Godfrey (Indiana University Extension, East Chicago, Ind.), and Ralph Eldon Lane (Flitcraft, Inc., Oak Park, Ill.).
University of Pennsylvania: Mr. Hyman N. Laden.
University of Rochester: Mr. Henry Pearce Atkins.
The College of St. Thomas: Mr. George Joseph Haltiner (of St. Paul).
Stanford University: Messrs. Charles Lester Clark and Stanley Eugene Rauch.
Syracuse University: Mr. Barnard Hinkle Bissinger.
University of Virginia: Mr. Franklin Guy Myers.
University of Washington: Misses Kathryn Eleanor Benson and Mary Elizabeth Layne.
Wellesley College: Professor Harriet Whitney Allen (Hollins College, Hollins, Va.).
University of Wisconsin: Messrs. Burton Houston Colvin, Wilhelm S. Erickson, William Grenfell Leavitt, and Clarence Bernhart Lindquist.

The University of Minnesota was elected to Institutional Contributing Membership in the Society.

Professor Francesco Tricomi of the University of Turin was admitted to the Society in accordance with the reciprocity agreement with the Unione Matematica Italiana.

The following appointments by President G. C. Evans were reported: as representative of the Society at the inauguration of Frank Jay Prout as President of Bowling Green State University on October 21, Professor J. B. Brandeberry; as representative of the Society at the inauguration of John Benjamin Magee as President of Cornell College on October 27, Professor E. W. Chittenden; as representatives of the Society at the Centenary Celebration of the American Statistical Association in Boston on November 27, Professor G. D. Birkhoff, and in Philadelphia on December 29, Professor S. S. Wilks; as representative of the Society at the inauguration of Homer Price Rainey as President of the University of Texas on December 9, Professor R. L. Moore; as a Committee on Arrangements for the 1940 Spring Meeting in Washington, D.C., Professor F. E. Johnston (chairman), Dr. Archie Blake, Professors T. R. Hollcroft, A. E. Landry, Florence M. Mears, E. W. Titt, and F. M. Weida; as a Committee on Arrangements for the Summer Meeting of 1940, Professors F. W. Perkins (chairman), W. D. Cairns, T. R. Hollcroft, Robin Robinson, L. L. Silverman, and H. L. Slobin; as a Committee on Arrangements for the Annual Meeting of 1940, Professors S. T. Sanders (chairman), W. L. Ayres, H. E. Buchanan, W. D. Cairns, W. V. Parker, and N. E. Rutt; as additional members of the Com-

mittee on Publicity, Mr. R. M. Foster and Dr. J. M. Thompson; as tellers for the election at the 1939 Annual Meeting, Professor F. R. Bamforth and Dr. E. R. Lorch; as auditors of the Society's books for the year 1939, Professors R. G. Archibald and A. E. Meder, Jr.; as the Committee on Nomination of Officers and Members of the Council for 1941, Professors Arnold Dresden (chairman), L. M. Graves, E. R. Hedrick, R. L. Moore, and J. L. Walsh.

The following representatives of the Society were appointed: on the Editorial Board of the Annals of Mathematics for a period of three years beginning with 1940, Professors T. H. Hildebrandt, Saunders MacLane, and G. T. Whyburn; on the Division of Physical Sciences of the National Research Council for a period of three years beginning July 1, 1940, Professor G. C. Evans; on the Council of the American Association for the Advancement of Science for the year 1940, Professors Arnold Dresden and J. R. Kline; on the advisory board of The Kosciuszko Foundation to assist Polish scholars, Professor J. R. Kline.

The Secretary reported that the ordinary membership in the Society is now 2,283, including 200 nominees of institutional members and 79 life members. There are also 87 institutional members. The total attendance of members at all meetings in 1939 was 1,354; the number of papers read was 489; the number of invited addresses was 10; the number of members attending at least one meeting was 841.

At the annual election which closed on December 27, and at which 406 votes were cast (no person not on the ballot receiving more than one vote for any office) (375 for the list nominated by the Council), the following officers were elected:

Vice Presidents, Dr. T. C. Fry and Professor F. D. Murnaghan.

Secretary, Dean R. G. D. Richardson.

Associate Secretaries, Professors W. L. Ayres and M. H. Ingraham, and Dean T. M. Putnam.

Member of the Editorial Committee of the Bulletin, Professor P. A. Smith.

Member of the Editorial Committee of the Transactions, Professor C. C. MacDuffee.

Member of the Editorial Committee of the Colloquium Publications, Professor J. D. Tamarkin.

Members of the Council, Professors L. E. Dickson, C. G. Latimer, N. H. McCoy, Saunders MacLane, and W. M. Whyburn.

The Secretary requested that the Society accept his resignation to take effect not later than the end of 1940.

The reports of the Treasurer and auditors (Professors R. G. Archibald and A. E. Meder, Jr.) showed a balance of $9,405.61 exclusive of the balances in the Colloquium, Sinking Fund, and special funds. The Society's investments including Endowment Fund, Sinking Fund, and other special funds had a market value on November 30, 1939, of approximately $150,000.00. The net interest income for the year was $5,377.25; dues from Institutional Memberships amounted to $6,530.00; dues from Contributing Memberships, $1,163.92; and dues from Ordinary Memberships were $13,541.71. The Colloquium Fund showed a balance of $8,075.38. The total received from the sale of the Society's publications was $11,534.24. The Board of Trustees adopted a budget for 1940 showing estimated expenditures and receipts of the Society (including Mathematical Reviews) as $53,800.00 and $40,800.00 respectively.

The Librarian reported that the Library of the Society now contains about 9,475 volumes.

The American Journal of Mathematics, which is published jointly by the Johns Hopkins University and the Society and to which the Society gives an annual subvention of $2,500, printed 1,008 pages during 1939.

An invitation from Dartmouth College to hold the 1940 Summer Meeting in Hanover, New Hampshire, was accepted, and the meeting was set for September 10–12. On invitation, the 1941 Summer Meeting is to be held at the University of Chicago in connection with its Semicentennial Celebration. There will be a meeting of the Society at Seattle, Washington, in June 1940, in affiliation with the American Association for the Advancement of Science. Times and places of additional meetings during 1940 were set as follows: February 24 and October 26 in New York, April 6 in Berkeley, December 30, 1940–January 1, 1941 in Baton Rouge.

On recommendation of the Committee on Special Membership Rates, the following addition to Article VI of the by-laws was recommended to the Society:

> Section 9. Any member who has been retired from active service on account of age and has been a member of the Society for twenty years or more may, upon notification to the Secretary of such retirement, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the Bulletin.

This change in the by-laws will be submitted to the Society for vote at the meeting on February 24, 1940.

A proposal from the Committee on the Disposition of Interest on Special Funds is to lie over for consideration until the next meeting

of the Council. This proposal involves the inauguration by the Society of a new series of books, somewhat smaller in size than books of the Colloquium Series and devoted chiefly to exposition of advanced topics.

Copies of the first number of Mathematical Reviews were on exhibition at the meeting. Eight hundred subscribers are already on the books. Three hundred fifty mathematicians, in all parts of the world, have accepted invitations to be reviewers. Arrangements are being made with various organizations for joint sponsorship of the journal with the Society.

A grant of one thousand dollars was announced from the Rockefeller Foundation for an experimental period of two years for the filming of all mathematical papers reviewed in the Mathematical Reviews.

The Emergency Executive Committee of the International Congress of Mathematicians which was to have been held in Cambridge, Massachusetts, in September 1940, announced that on December 20, 1939, notice was sent to all persons interested that the Congress has been postponed.

It was decided that no Josiah Willard Gibbs Lecture would be given during the year 1940.

Professor R. L. Wilder was invited to give the Colloquium Lectures in 1942. It was announced that Professor G. T. Whyburn had been asked to give his Colloquium Lectures in 1940. A book entitled *Lattice Theory* by Professor Garrett Birkhoff was accepted for publication in the Colloquium Series.

The Transactions editors announced that space is now available promptly in that journal. Professor E. T. Bell's term having expired, Professor Saunders MacLane has been appointed associate editor in his place.

Professor M. H. Stone proposed that a letter be sent to sister mathematical organizations in various parts of the world, suggesting that mathematicians use their influence to conserve the scientific resources of the world against the day when peace shall reign once more and to mitigate in every way possible the horrors of war. This proposal is to be presented to the Society for its action at the February meeting.

Titles and cross references to the abstracts of papers read at the sessions follow below. The papers were read as follows: papers numbered 1 to 6 in the section for Analysis on Tuesday afternoon; papers 7 to 12 in the section for Topology on Tuesday afternoon; papers 13 to 20 in the general session on Wednesday morning; papers 21 to 28

in the section for Analysis on Wednesday afternoon; papers 29 to 35 in the section for Algebra on Wednesday afternoon; papers 36 to 40 in the section for Logic and the Theory of Sets on Wednesday afternoon; papers 41 to 49 in the section for Applications on Thursday afternoon; papers 50 to 55 in the section for Topology on Thursday afternoon; papers 56 to 62 in the section for Analysis on Friday morning; and papers 63 to 102, whose abstract numbers are followed by the letter *t*, were read by title. Professor H. W. Smith was introduced by Professor O. H. Hamilton, Dr. Stefan Bergman by Professor W. T. Martin, Dr. M. H. Heins by Professor J. L. Walsh, Dr. D. C. Spencer and Dr. Samuel Eilenberg by Professor W. L. Ayres, Dr. A. T. Brauer by Professor Hermann Weyl, Dr. Alexander Wundheiler by Professor D. J. Struik, Professor André Gleyzal and Dr. M. S. Hendrickson by Professor Henry Blumberg, Dr. A. E. Heins by Professor A. H. Smith, and Dr. W. A. Patterson by Professor Lincoln LaPaz. Paper 4 was presented by Dr. J. J. Eachus, paper 8 by Dr. P. V. Reichelderfer, paper 12 by Professor E. W. Miller, paper 28 by Professor Marston Morse, paper 42 by Dr. Eric Reissner, paper 48 by Professor Norbert Wiener, and paper 55 by Dr. D. W. Hall.

1. Mark Kac: *On the partial sums of the exponential series.* (Abstract 46-1-90.)

2. Norman Levinson: *A proof of Hardy's theorem on the zeros of the zeta function.* (Abstract 46-1-96.)

3. D. G. Bourgin: *The Dirichlet problem for the damped wave equation.* (Abstract 46-1-49.)

4. R. J. Duffin and J. J. Eachus: *Sets invariant under a polynomial operator.* (Abstract 46-1-63.)

5. Otto Szász: *On strong summability of Fourier series.* (Abstract 46-1-127.)

6. Alfred Korzybski: *General semantics: extensionalization in mathematics, mathematical physics, and general education.* III. *Over/under defined terms.* (Abstract 45-11-424.)

7. A. D. Wallace: *Relatively non-alternating transformations.* (Abstract 46-1-131.)

8. Tibor Radó and P. V. Reichelderfer: *On cyclic transitivity.* (Abstract 46-1-30.)

9. J. W. T. Youngs: *A remark on cyclic transitivity.* (Abstract 46-1-136.)

10. O. G. Harrold (National Research Fellow): *Exactly* $(k, 1)$ *transformations on linear graphs.* (Abstract 46-1-76.)

11. G. E. Schweigert: *A local property arising from certain interior transformations.* (Abstract 46-1-120.)

12. Samuel Eilenberg and E. W. Miller: *On 0-dimensional upper-semi-continuous collections.* (Abstract 46-1-67.)

13. R. D. James: *A problem analogous to the problem of prime pairs.* (Abstract 46-1-85.)

14. A. A. Albert: *On p-adic fields and cyclic algebras.* (Abstract 46-1-44.)

15. W. J. Trjitzinsky: *Some problems in the theory of singular integral equations.* (Abstract 45-11-438.)

16. G. B. Price: *On the theory of integration.* (Abstract 46-1-114.)

17. I. J. Schoenberg: *On metric arcs of vanishing Menger curvature.* (Abstract 46-1-119.)

18. R. P. Boas: *Some uniqueness theorems for entire functions.* (Abstract 46-1-48.)

19. Karl Menger: *On Cauchy's theorem in the real plane.* (Abstract 46-1-104.)

20. Norbert Wiener: *A canonical series for symmetric functions in statistical mechanics.* (Abstract 46-1-133.)

21. H. W. Smith: *The oscillation of solutions of the differential boundary value problems of the fourth order.* (Abstract 46-1-125.)

22. C. E. Sealander: *Some third order irregular boundary value problems.* (Abstract 46-1-122.)

23. Stefan Bergman: *On the approximation of functions satisfying a linear partial differential equation.* (Abstract 46-1-46.)

24. J. W. Calkin: *Semi-bounded forms and self-adjoint boundary value problems.* (Abstract 46-1-52.)

25. M. H. Heins: *On the conformal mapping of a multiply-connected region into itself.* (Abstract 46-1-78.)

26. D. C. Spencer: *On mean one-valent functions.* (Abstract 46-1-126.)

27. Abe Gelbart: *On the growth of a function of two complex variables given by its power series expansion on certain hypersurfaces.* (Abstract 46-1-71.)

28. Marston Morse and C. B. Tompkins: *Minimal surfaces of unstable type.* (Abstract 46-3-149.)

29. D. T. Sigley: *k-set groups.* (Abstract 45-11-436.)

30. B. W. Jones: *Related genera of quadratic forms.* Preliminary report. (Abstract 46-1-87.)

31. C. G. Latimer: *On a certain equation.* (Abstract 46-1-95.)

32. R. P. Dilworth: *Ideals in Birkhoff lattices.* (Abstract 46-1-61.)

33. Max Zorn: *Alternative rings with nilpotent elements.* (Abstract 46-1-137.)

34. A. T. Brauer: *On the density of the sum of sets of positive integers. II.* (Abstract 46-1-50.)

35. R. F. Clippinger: *On matrix products of positive powers of given matrices.* Preliminary report. (Abstract 46-1-55.)

36. A. H. Diamond: *Postulates for the theory of strict implication.* (Abstract 46-1-60.)

37. Walter Jennings: *Some implications of the continuum hypothesis.* (Abstract 46-1-86.)

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99. A. D. Hestenes: *On certain matrix ϕ-difference equations.* Preliminary report. (Abstract 46-1-82-*t*.)

100. Deane Montgomery and Leo Zippin: *A Hilbert axiom for topological transformation groups of space.* (Abstract 46-1-107-*t*.)

101. W. C. Graustein: *Harmonic minimal surfaces.* (Abstract 46-3-142-*t*.)

102. E. D. Hellinger: *Classes of monotone functions.* Preliminary report. (Abstract 46-3-145-*t*.)

W. L. AYRES,
Associate Secretary

BOOK REVIEWS

Elementary Number Theory. By J. V. Uspensky and M. A. Heaslet. New York and London, McGraw-Hill, 1939. 10+484 pp.

Numerous historical references and applications of the theory presented, as well as detailed proofs, make this book especially suited to novices in the field. As noted in the introduction "owing to self-imposed limitations in the size of the book, many topics of interest had to be omitted." Analytical and geometrical methods have been avoided, while topics such as continued fractions and integral transformations of forms have been omitted entirely. On the other hand the book covers with thoroughness various fundamental problems which have inspired much of the research in this field. The results are illustrated by numerous examples, some solved in the text, others left to the reader.

The first five chapters are devoted to various problems which can be readily treated without the use of congruences. In fact, the notion of congruence is deferred until the sixth chapter when the reader has become familiar with a number of the fundamental concepts of number theory. The first chapter contains a discussion of elementary properties of integers, a method of computing polygonal numbers, and a discussion of scales of notation. It also contains an analysis, in terms of binary representations of numbers, of the generalized Chinese game of Nim. In the second chapter there is a treatment of common divisors and multiples, and a solution of the Diophantine equation $x^2+y^2=z^2$ in integers. The third chapter contains Lame's theorem, Euclid's least remainder algorithm, and application of this algorithm to the solution of linear Diophantine equations. The fourth chapter is devoted to a discussion of prime numbers. The topics considered are the sieve of Eratosthenes, the unique factorization theorem, the number and sum of divisors of an integer, perfect numbers, Mersennes's numbers, and the distribution of primes. In the fifth chapter there is a discussion of relative primeness, and in particular of Euler's function $\phi(n)$ and Moebius's function $\mu(n)$, with results based on a well known combinatorial theorem. This theorem is applied to the problem of determining the number of primes less than a given integer, and the chapter closes with Meissel's formula for this number.

After the introduction of the concept of "congruence," Chapter 6 is devoted to Fermat's theorem concerning the congruence $a^{p-1}\equiv 1 \pmod{p}$ and the Euler generalization on $a^{\phi(m)}\equiv 1 \pmod{m}$. Several

proofs are given for Fermat's theorem, followed by Gauss's proof of Wilson's theorem. Chapter 6 is followed by an appendix on the problem of constructing magic squares with the solution based on the notion of congruence. In Chapter 7 the authors discuss the number of roots of polynomial congruences of arbitrary degree with application to the theory of residues of lth powers of a prime p. First degree Diophantine equations treated in the third chapter are solved here by a little known method due to Voronoi. The chapter closes with the Chinese method for solving sets of linear congruences in one unknown. The theory of linear congruences is applied in an appendix on calendars. Chapter 8 is devoted to residues of powers with special emphasis on primitive roots and indices, and there is an application to the solution of congruences. The concept of "exponent to which a number a belongs modulo m" is applied in an appendix on card shuffling. In Chapter 10 the authors take up various aspects of the theory of quadratic residues. The chapter is concerned primarily with two fundamental problems: Is a number a a quadratic residue or non-residue of a given prime p, and for what prime moduli is a given number a a quadratic residue or non-residue? The first problem is solved by means of a criterion of Euler. For $a=-1$, 2 and -2 the second is solved by elementary direct methods. Legendre and Jacobi's symbols are introduced, and the corresponding quadratic reciprocity laws are proved. The chapter ends with the solution of general quadratic congruences.

In Chapter 11 the authors develop some special aspects of the theory of quadratic forms. The chapter begins with a discussion of Fermat's equation $t^2-au^2=1$, wrongly attributed by Euler to Pell, and the generalization $x^2-ay^2=m$ of this equation. The results are applied to determine conditions that a given number m be prime. The chapter contains Kummer's proof of the reciprocity law based on Fermat's equation, and Dickson's proof that each integer is a sum of four squares. In Chapter 12 results of earlier chapters are applied to special Diophantine equations, such as $x^2+ay^2=z^m$, $x^2+c=y^3$. This is followed by a proof that the equation $x^4+y^4=z^2$ has no solution in integers none of which is zero, a proof accomplished by the standard technique of showing that if $x^4+y^4=z^2$ has a solution (x_1, y_1, z_1), this equation has a solution (x_2, y_2, z_2) with $z_2<z_1$. The final chapter (Chapter 13) is devoted to the general arithmetical identities proved by Liouville by elementary methods. These identities are used to prove that the primes of the form $8k+1$ or $8k+3$ are sums of a square and the double of a square, and are also employed to give an elementary proof of Jacobi's result on the number of repre-

sentations of an integer as a sum of four squares. Finally, a proof, based on these identities, is given for the theorem of Gauss that all integers except those of the form $4^k(8N+7)$, $k \geqq 0$, are representable as a sum of three squares. The idea of this proof goes back to Kronecker.

An exception to the other chapters of the book which form a closely knit and well-integrated unit is Chapter 9 on Bernoulli numbers. This chapter is independent of the rest of the book except for a minor reference to it later. The chapter treats some fundamental properties of Bernoulli numbers. In particular, the authors give Staudt's theorem on the fractional parts of Bernoulli numbers, and use a theorem of Voronoi to yield information on the types of factors which occur in the numerators and denominators of Bernoulli numbers. The chapter ends with Kummer's congruence on Bernoulli's numbers. There are no applications.

Although the style is clear and detailed, more clarity would have been obtained if there had been uniformity in the statement of the theorems and corollaries. Sometimes the theorems and corollaries are incorporated in the text, with a part of the proof, while in other places the theorems and corollaries stand out in paragraphs by themselves. In the treatment of roots of polynomial congruences one misses the important notion of a "field of numbers," since some of the results, such as that a congruence of degree n with prime modulus has no more than n incongruent roots, are merely theorems valid for any field of numbers and in particular for modular fields. The use of the concept of field, so lacking in books on number theory, would emphasize the importance of many of the problems considered. Some number theorists might take exception to the statement, pages 19–20, "The theory of numbers, unlike some other branches of mathematics, is a purely theoretical science without practical applications." Although certainly deficient in this sense compared with other fields of mathematics, such problems as the calendar problems considered in this text seem to fall in the class of practical applications. Historians might take exception to the statement, page 20, "Pierre de Fermat (1601–1665) was the first man to discover really deep properties of numbers."

We mention a few errors that came to our attention. In line 2 from the bottom of page 18 the following phrase "$r_3+s_3=4 \leqq k$, so that we adjoin the number 3, and change the units in column l_3 to 0's" should read "$r_3+s_3=4>k$, whence we change one zero in column l_3 to 1." On page 19 the columns (which we write as rows) *, 0, 1→0, 1→0, 1→0, 0, 0 and 0110<1110, 0110<1101, 0100<1001, 10<11,

100, 10, /440, should read *, 0→1, 1, 1, 1, 0, 0 and 0111<1110, 0111<1101, 0101<1001, 11, 100, 10, /444, respectively. The expression "$n \geqq 4$," line 7, page 89, should read "$10 > n \geqq 4$." In line 15, page 100, the symbol $[n/]$ should be $[n/p]$. On page 194, line 2, the phrase "modulo p^{a-1}" should follow "is congruent." On page 299 the symbol $(p_1 q)$ in the 9th line from the bottom should read (p_1/q), and $[(p_k-1)/2]$ in line 3 from the bottom should read $[(p_i-1)/2]$.

On the whole the authors have succeeded in presenting number theory in a fascinating and highly instructive manner. Their work will fill an important place in the literature.

RUFUS OLDENBURGER

Differential Geometrien in den Kugelräumen. Vol. 1. *Konforme Differentialgeometrie von Liouville und Möbius.* By T. Takasu. Tokyo. Maruzen, 1938. 18+457 pp.

This handsome textbook gives a comprehensive account of the differential geometries in spaces in which spheres are taken as spatial element. It is the outgrowth of a series of fundamental investigations in which the author has been engaged since 1924, and which have, so far, been laid down in 47 papers. He has also used, in the widest and most generous sense of the word, the work of other authors, which he lists in an extensive and probably complete bibliography of 202 titles. This book is, therefore, the most authoritative source on the subject, and the reader will at the same time be pleased with the variety and beauty of the results and amazed at the thoroughness of the work.

The main body of research already has been published in various "Science Reports of the Tôhoku Imperial University," beginning in March, 1928. The present volume often follows the previous publications, but also departs from it, giving more material and more references. The author, in the preface, gives his own classification of the different geometries as derived from Lie's geometry of the spheres, embracing several types of conformal geometry, including that of Laguerre, of affine, and of non-euclidean geometry. We repeat his fundamental principle as follows:

"1. The conformal space of Möbius is a non-euclidean space with movable absolute spheres, hence also with variable curvature.

"2. The space of Laguerre is a euclidean space with movable absolute circles, hence also with variable unit of angular measurement.

"3. The space of Lie is a conformal space with a movable absolute complex of spheres and at the same time a dual-conformal space

with movable dual-absolute complex of spheres, so that the unity of measure of length and angle are functions of position."

These three principles, according to the author, allow a perfectly clear interpretation of the analytical construction of these geometries, and one which has no gaps. He believes that this elucidates completely a field which has had too little attention.

The geometry of spheres has, indeed, been little investigated and mostly in an unsystematic way. Pentaspherical and hexaspherical coordinates, which are the appropriate types of coordinates for such geometries, were originally only used by Darboux (1889) and Coolidge (1916), until after 1922 when a new series of investigations began, with which the names of Vessiot, Thomsen, Blaschke, Liebmann and Kubota are connected, of whom some also use cartesian coordinates. There is a close relation in method between this book of Professor Takasu and the third volume of Blaschke's *Differentialgeometrie*, written by Thomsen, and which appeared in 1928. There is also a considerable difference between the books, Thomsen's book being only partly devoted to differential geometry, Takasu investigating in a very systematic way only differential geometry.

This book of 457 pages is only the first volume of a planned set of three. It deals with the geometries of Liouville and Möbius. The second volume will give the differential geometry of Laguerre, the third volume that of Lie.

The book consists of two "Abschnitte," a short first one of twenty-five pages giving introductory theories, explaining the relation of the three geometries of the sphere, the second one bringing the theory of systems of ∞^1 circles, of nets of curves, of systems of ∞^1 and ∞^2 spheres, of systems of ∞^1 circles in space, and of triply orthogonal systems. It follows closely the classical exposition of ordinary differential geometry. The systems of ∞^1 circles correspond to the curves in the plane (taken as envelopes of straight lines), the systems of ∞^1 spheres to curves in space, the systems of ∞^2 spheres to surfaces, the systems of ∞^1 circles in space to ruled surfaces, the correspondence is with non-euclidean geometry rather than with euclidean because of the nondegenerated quadratic relations between the coordinates.

It is impossible to mention even a small part of the rich material. We have generalizations to conformal space of Bertrand curves, of the four vertex theorem, of Meusnier's and Euler's theorems. Many classical results receive for the first time their proper setting. Dualities are often shown by comparison of theorems and proofs on the right and left side of the page. The systematic exposition can often reframe an old proof with the new and unified methods in the new

conformal language and notation. There is a certain finality about the content which makes it the foundation of every further study on the subject.

It is not always easy to follow the author and we would like to ask him to be kind with his readers when he gives the final touches to the other two volumes. As an example, let us take the beginning of the book. On page 1 we hurry immediately into the midst of things. An oriented sphere is defined by 5 pentaspherical coordinates connected by a quadratic relation. The meaning of these coordinates is not explained, we must take this from other books, and the quadratic relation is only given as $(yy)_5=1$, which leaves it to the reader to discover what it means. We are not informed either, whether the coordinates are ordinary or general pentaspherical coordinates, and have to discover this later, from the context. On page 2 we read that the different projective orientation processes must be understood in the sense of the author's non-euclidean geometry. This is an essential point, and we must therefore first go to volume 26 of the Tôhoku Mathematical Journal (1926) to find what it means. The same thing happens on page 3, where we are referred to another paper to find the meaning of certain equations expressing a doubly oriented sphere. Such difficulties could easily be avoided if the author, at the beginning, would not presuppose more than an average college knowledge of pentaspherical coordinates, projective and non-euclidean geometry, as, for example, Blaschke, Thomsen, or Fubini-Cech have done in their related expositions. We are sure that Professor Takasu will only do justice to his beautiful investigations if he can agree to such modifications in his presentation.

D. J. STRUIK

Superficie Razionali. By Fabio Conforto. Bologna, Zanichelli, 1939. 16+549 pp.

Although there is an extensive literature on rational surfaces, it is scattered through the periodicals in various languages, and the methods of proof differ widely as the theory gradually develops through more than a century. On the other hand, a knowledge of this field is indispensable to the study of algebraic geometry of more than two dimensions, and to some phases of analysis.

The purpose of the present book is to supply this need of a systematic development of the subject from the present point of view, starting at the beginning and providing all the necessary details of the general theory, but referring to original papers for further

perusal of particular features. A knowledge of plane geometry, including plane Cremona transformations and algebraic plane curves is presupposed. Throughout the book the guiding principle is that of continuity of the parameters in a linear system, the methods of proof following closely those of the four volume treatise of Enriques and Chisisni[1] to which frequent reference is made.

The volume is divided into two parts, the first being concerned with rational surfaces of order less than 5, developed largely along traditional lines, and the other to the systematic theory. But this first part includes one feature that deserves special mention. After discussing those quartic surfaces that have a double curve, the book provides an exhaustive analysis of all the quartic surfaces that are rational and have just one isolated singular point. Here, in particular, the principle of continuity of the parameter in a linear system is employed in such a way that a limiting value of the parameter defines a surface with a singularity of the desired form. It is clearly shown that the threee types of Noether quartics correspond to the three types of plane involutions of order two, the images, on the surface, of a pair of conjugates on the plane, being a pair of residual points on a line through the double point. Other features in this first part are the exhaustive discussion of all possible cases, including the Steiner surface and Cayley's cubic scroll.

The second part begins with surfaces having a pencil of rational curves, then those having only rational plane sections, followed by elliptic, hyperelliptic ($p=2$) and those of genus 3, not hyperelliptic. In each case, the characteristic properties of the representative net of plane curves are established. Apparent digressions are the classifications of plane involutions of order two and the conditions for rationality of a double plane, but both results are shown to be necessary to insure that the classification of rational surfaces is complete.

A long chapter is devoted to other rational surfaces; it includes the quintics having a multiple curve, but does not claim to be exhaustive. The last chapter contains a complete treatment of the theorem of Castelnuovo that all plane involutions, of any order n, are rational. In the main, the proof in the original paper is reproduced, but important simplifications are achieved by systematic use of continuity in linear systems.

Frequent and extensive references to the literature are made, and a convenient index is provided, but much of the recent literature is not included, especially that of non-European origin; moreover, many

[1] *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, Bologna, Zanichelli, 1915, 1918, 1923, 1934.

of the earlier theorems are not referred to their sources, but to later proofs of them given in works supposed to be more accessible.

The writing and proofreading have been done with great care; apart from a few trivial errors in orthography, the only error is the reference to G. Kantor, footnote on page 131, instead of S. Kantor.

This book will be welcomed by workers in algebraic geometry; it competently fills a gap in the preparatory literature.

VIRGIL SNYDER

Modern Science, A Study of Physical Science in the World Today. By Hyman Levy. New York, Knopf, 1939. 736 pp. 160 illustrations.

Professor Levy's ability to present general relationships and abstractions in an interesting simple way has been commended by reviewers of his previous recent book "A Philosophy for a Modern Man." This faculty is again evident in his analysis and evaluation of social and intellectual forces in the development of physical science.

Part I of the book deals with the background of social life within which science has developed as one among many "channels" of human energy. In an interesting discussion Professor Levy contends that the work of men like Newton was largely a consequence of commercial and other social factors rather than a spontaneous intellectual activity.

Part II is concerned with the nature, methods and unity of science. Considerable stress is laid on the common occurrence of sequences of phases separated by discontinuities in physical processes. Many usually overlooked examples of these phases are given.

Parts III, IV and V deal with mathematical symbols and physical (theoretical) models. Some general algebraic and geometrical concepts are discussed. The idea of limits is presented in a remarkably concrete way. The rigor is surprisingly good for a popular discussion, but there are some undesirable implications, for example, termwise differentiability of infinite series is accepted without question. An instructive treatment of mass, momentum, impulse and energy is given. Aeronautical science is used to illustrate the unity of theory and experiment. The development of non-euclidean geometries is used to illustrate how science can "shake off the past."

Part VI is rather discursive in its discussion of astronomy, geology, atomic theory and relativity. The treatment of mountain building does not include the more modern theories of Joly and others.

Part VII on the "Age of Light" uses the history of methods of illumination and their social consequences to illustrate the depend-

ence of various branches of science and human activity on one another.

While the reviewer can recommend this book as interesting, clear and stimulating for the moderately well informed reader, he feels that it is unnecessarily long due to redundance of examples, and that its lack of references is an unfortunate weakness.

J. K. L. MacDonald

Punktreihengeometrie. By E. A. Weiss. Leipzig and Berlin, Teubner, 1939. 8+232 pp.

The purpose of this book is to serve as an introduction to the projective geometry of higher dimensional spaces. The author adopts the point of view of Reye in regarding such space as the map space of various three-dimensional configurations and he thereby succeeds in bringing together a large variety of topics.

After a brief recapitulation of the elementary geometry of the line, the author introduces the concept of a (linear) point range (punktreihe) on a line obtained by equating to zero a bilinear form in variables (ξ_1, ξ_2) and (τ_1, τ_2), which he writes using the Clebsch-Arnold symbolic notation as $(\gamma\xi)(\mu\tau)=0$. He interprets (τ_1, τ_2) as a parametrization of the points (ξ_1, ξ_2) of the line, and he subsequently defines a point range more generally as a "one-dimensional rational manifold provided with a definite parametrization." By means of the coefficients $\gamma_i\mu_j$ linear point ranges on the line can be put in 1-1 correspondence with the points of projective 3-space; singular ranges map into a ruled quadric, pencils and bundles of ranges into lines and planes. From the properties of singular ranges the author deduces the elementary properties of the quadric.

The two main chapters of the book deal with point ranges (and their duals) in two and three dimensions. By using symbolic notation, the Clebsch correspondence principle, and similar devices, the author derives easily such fundamental results as the harmonic properties of a quadrilateral, the projective generation of a conic, and the polar theory in the plane. Further properties of a conic follow by considering it as a point range of the second order defined analytically by $(um)(\gamma\tau)^2=0$. Pascal's configurations are obtained by mapping binary quadratic forms on the points of a plane.

Linear point ranges in the plane can be mapped on R_5, projective space of five dimensions, and singular ranges correspond to a Segre manifold of three dimensions and of order 3. The singular ranges of a pencil or bundle will map into rational cubic curves and surfaces.

The third chapter deals with geometry in R_3 and the development

parallels that of the second. Now the map space of linear point ranges is an R_7, the Segre manifold is quartic and four-dimensional, and properties of quartic curves and surfaces follow by considering the singular ranges in pencils and bundles. Here, as in the preceding chapters, the author studies the collineations of the map space which leave fixed the corresponding Segre manifold. By noting that quadric hypersurfaces in R_7 are the maps of linear complexes, the author leads naturally to the principle of triality of Study and Cartan, null systems, the line sphere transformation, and properties of oriented spheres.

The final chapter deals with trilinear forms and some applications to the preceding situations as well as to non-euclidean geometry.

The author has succeeded in bringing together a large variety of geometrical configurations and studying them in a way which will stimulate the student's interest and arouse his admiration for geometrical methods.

HARRY LEVY

Mathematical Recreations and Essays. 11th edition. By W. W. Rouse Ball. Revised by H. S. M. Coxeter. New York, Macmillan, 1939. 16+418 pp.

For almost half a century the earlier editions of this book[1] have provided a rich supply of mathematical topics commonly known as recreations, which although they often involve fundamental mathematical methods and notions, yet make their appeal in the spirit of a game or a puzzle, rather than with an eye to the usefulness of their conclusions. Since no knowledge of the calculus or of analytic geometry is presupposed, many of these recreations make excellent subjects for student talks in undergraduate mathematics clubs.

The eleventh edition, revised by H. M. S. Coxeter after Ball's death, not only "aims to preserve the spirit of Ball's delightful book," but does. It is, nonetheless, a thorough-going revision of the tenth edition. The chapters on Mechanical Recreations, Bees and their Cells, and String Figures have been omitted. In their stead we find the following material.

(1) A large new section of arithmetical recreations (chap. II), in-

[1] Editions of this book have also appeared in French and Italian. A three-volume edition in French with considerable additional material, especially in the history of numbers, was edited by J. Fitz-Patrick in Paris (1907–1909), and a new edition appeared in 1926–1927 with new material by A. Margossian, Reinhart, J. Fitz-Patrick and A. Aubry. Two Italian editions were printed in Bologna, the first in 1911 by D. Gambioli, 398 pp., and the second in 1927 by D. Gabioli and G. Loria.

cluding discussions of derangements of n quantities, repeating decimals, rational right-angled triangles, finite arithmetics, and the distribution of primes; (2) a new chapter (chap. V) on polyhedra—a subject in which the author is expert—written in a manner which is intelligible to a beginner equipped with a modicum of ability in space perception, and supplemented by two excellent plates showing models of the regular and semiregular solids; (3) a more extended treatment of magic squares and their generalizations; (4) a discussion, in the chapter on map colouring problems, not only of the famous four-colour problem in the plane, but of the seven-colour problem on the torus, and of other topics connected with the stereographic projections of the regular polyhedra.

The first edition of 1892 was entitled "Mathematical Recreations and Problems" and consisted of two parts, the one on Recreations and the other on Problems and Speculations. Of the five chapters in the latter part only the one on Three Classical Problems has been retained in all editions. The chapters on Astrology, Hyper-space, Time and its Measurement, and the Constitution of Matter, became the basis for a much enlarged second half of the fifth edition (1911) which bore the title of Mathematical Essays. The space devoted to essays was considerably reduced in the tenth edition (1922). It now includes the Three Classical Geometrical Problems (chap. XII), an essay on calculating prodigies (chap. XIII), and an essay on cryptography (chap. XIV), the latter being completely revised for the present edition by Abraham Sinkov, a cryptanalyst in the U. S. War Department.

A partial summary of the contents of the book is as follows. *Chapter I* includes arithmetical recreations whose interest is mainly historical rather than arithmetical. Some of these are of the "think of a number" type, others involve digit notations, and still others are tricks with cards or games with counters. *Chapter II* opens with a series of arithmetical fallacies, continues with problems of probability derangements and arrangements, decimal expansions, rational triangles, finite arithmetics, D. H. Lehmer's number sieve for prime factors, and concludes with a discussion of perfect numbers, Mersenne's numbers, and Fermat's theorem. *Chapter III* is composed mainly of geometrical fallacies and paradoxes, problems in dissection, cyclotomy and area-covering. The deltoid solution to Kakeya's minimal problem, erroneously attributed to Kakeya (p. 100) was really suggested by Professors Osgood and Kabotą according to Question 39, American Mathematical Monthly (1921), p. 125. *Chapter IV* is concerned with statical and dynamical games of position.

Among topics discussed are some extensions of the game of three in a row, tessellations of the plane, problems with moving counters, and the effect of cutting a Möbius strip in various ways. *Chapter V* gives a comprehensive elementary discussion of the relations between the faces, edges, and vertices and the associated angles of the regular solids and the Archimedean solids, which is well illustrated by good figures. Stellated polyhedra, solid tessellations, and the kaleidoscope each receive some attention. The use of the term Platonic for the regular solids might be questioned since they were known before Plato. *Chapters VI* and *VII* contain familiar recreations associated with the chessboard and with magic squares. Similar problems with dominoes and with magic cubes are also discussed. *Chapter VIII* treats the general theory of the four-colour problem more elaborately than the earlier editions of this book, mentions briefly such matters as orientable surfaces and dual maps, and more fully the seven-colour mapping problem on the torus, and finally considers various colouring problems on the regular polyhedra. *Chapter IX* discusses mazes and other similar problems whose solutions depend on the unicursal tracing of a route through prescribed points (nodes) over various given paths. *Chapter X* features certain combinatorial problems known under the title of Kirkman's school-girl problems, and ends with a similar problem about arranging members of a bridge club at tables so that different members shall play together in successive rubbers. *Chapter XI*, on Miscellaneous Problems, contains an account of the Fifteen Puzzle, the Tower of Hanoï, Chinese Rings, and various mathematical card tricks. *Chapter XII* contains the famous classical problems concerning the duplication of the cube, trisection of an angle, and quadrature of the circle. *Chapter XIII* is an essay on calculating prodigies which introduces over a dozen famous mental calculators beginning with Jedediah Buxton and Thomas Fuller in the eighteenth century, and including two American calculators Zerah Colburn and Trueman Henry Safford, and gives something of their histories and the type of problems they could solve. *Chapter XIV* is a chapter on cryptography and cryptanalysis written by Dr. Abraham Sinkov. It presents in easily understandable form the chief elements in a cryptographic system, and gives various possible ways for attempting to solve such a cipher.

J. S. FRAME

Einführung in die Zahlentheorie. By Arnold Scholz. (Sammlung Göschen, no. 1131.) Berlin, de Gruyter, 1939. 136 pp.

The topics which should be taken up in an introduction to the

theory of numbers do not seem to have been well standardized. Each author discusses a certain minimum set of topics and then proceeds to whatever subject interests him. In this book the choice of further topics is well taken and quite varied.

After an introduction to the series of natural numbers the author takes up the subjects common to most such books. Included here is a descriptive section on the distribution of prime numbers in which a number of interesting numerical examples are given. The later part of the book takes up the representation of integers by binary quadratic forms in considerable detail and it includes a chapter on the numerical solution of various problems that arise in number theory.

The principle followed in the preparation of this book seems to have been conciseness. This is apparent, not only in the typography, but even in the author's manner of writing. The earlier chapters have been written with care but this brief manner of writing makes the later chapters more difficult to read.

Because of its conciseness this book seems more suitable as supplementary reading rather than as a first introduction to the theory of numbers.

H. S. ZUCKERMAN

Anwendung der Eulerschen Reihentransformation zur Summierung der Dirichletschen Reihen, der Fakultätenreihen und der Newtonschen Reihe. By N. Obreschkoff. Berlin, Verlag der Akademie der Wissenschaften, 1938. 36 pp.

This pamphlet is a reprint of an article published in the Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalish-mathematische Klasse, 1938. The author applies the Euler-Knopp method of summability to Dirichlet series, to factorial series and to Newton's series, obtaining results which show that this summability method is very useful in such cases and that it is ordinarily more powerful than the methods of summation of Cesàro and of Riesz.

The ordinary Dirichlet series $\sum a_n/(n+1)^s$ is first investigated. The author shows that if this series is E_k-summable for $s=s_0$, then it is E_k-summable for every s for which $R(s)>R(s_0)$, and he gives a formula for the generalized sum there. Use is made, in the proof, of the fundamental Silverman-Toeplitz conditions for summability. A formula is derived for the abscissa of E_k-summability in terms of the E_k-transform of the series $\sum a_n$. Corresponding results are obtained for absolute E_k-summability.

Similarly, it is shown that if the factorial series $a_0/s+\sum n!a_n/s(s$

$+1)(s+2)\cdots(s+n)$ is E_k-summable for $s=s_0$, then it is E_k-summable for every s for which $R(s)>R(s_0)$, and a formula is given for the sum. Also, a formula is derived for the abscissa of E_k-summability. Extensions are made to absolute E_k-summability. Similar results are obtained for the Newton series $a_0+\sum(-1)^n a_n(s-1)(s-2)\cdots(s-n)/n!$.

LLOYD L. SMAIL

NOTES

A Conference in Topology will be held at the University of Michigan June 24–July 6, 1940.

Mr. L. L. Locke has placed his collection of old calculating machines in the Smithsonian Institution. This collection contains many unique and interesting machines, among them the first two machines made in the United States and the first direct multiplication machine.

A mathematics teaching seminar on a post-doctoral level is being conducted this year at Reed College, with Professor F. L. Griffin in charge. The four fellows who are participating are: Mr. H. E. Goheen, Dr. L. Louise Johnson, Mr. R. A. Rosenbaum, and Dr. Henry Scheffé.

Professor P. A. M. Dirac of the University of Cambridge has been awarded a medal by the Royal Society, London, for his work in the new quantum mechanics.

The University of London has conferred the degree of doctor of science on Dr. Niels Bohr, director of the Institute of Theoretical Physics at the University of Copenhagen.

Professor V. V. Narlikar of Benares Hindu University has been elected a fellow of the National Institute of Sciences of India.

The Institute of Mathematical Statistics has elected Associate Professor S. S. Wilks of Princeton University as president, Associate Professor A. T. Craig of the University of Iowa and Associate Professor C. C. Craig of the University of Michigan as vice presidents, and Professor P. R. Rider of Washington University as secretary-treasurer.

Professor A. B. Coble of the University of Illinois has been elected vice president of the American Association for the Advancement of Science. Professor J. L. Walsh of Harvard University has been elected a member of the Sectional Committee for Mathematics.

Professor L. E. Dickson of the University of Chicago has been named Eliakim Hastings Moore Distinguished Service Professor Emeritus of Mathematics.

Professor William Gillespie of Princeton University has been given the title of professor emeritus.

Professor Gabriele Mammana of the University of Naples and Pro-

fessor Luigi Sobrero of the University of Rome have been appointed to professorships at the University of Rio de Janeiro.

Assistant Professor R. V. Blair of Vanderbilt University has been promoted to an associate professorship.

Associate Professor E. R. Breslich of the University of Chicago has retired.

Assistant Professor R. C. Bullock of North Carolina State College has been promoted to an associate professorship.

Dr. A. S. Gale of the University of Rochester has resigned as dean of the College for Men. He will continue as chairman of the department of mathematics.

Associate Professor A. E. Gault of Bradley Polytechnic Institute has been promoted to a professorship.

Mr. Coleman Herpel of Pennsylvania State College at Hazleton has been promoted to an assistant professorship.

Dr. W. A. Patterson of Fenn College has been promoted to an assistant professorship.

Professor G. Y. Rainich of the University of Michigan will be on leave for the second semester.

Professor T. M. Simpson of the University of Florida has been appointed acting dean of the graduate school.

Reverend J. P. Smith of St. Peters College, Jersey City, has been appointed to an associate professorship at Georgetown University.

The following appointments to instructorships are announced: University of California: Dr. J. L. Brenner; University of Detroit: Dr. Alvin Sugar; Georgia School of Technology: Dr. Nelson Robinson; University of Illinois: Dr. I. M. Niven; Johns Hopkins University: Dr. Herbert Busemann; University of Maine: Mr. A. W. Jones; University of Maryland: Dr. F. J. Weyl; Scranton-Keystone Junior College: Mr. B. J. Trapani; Wayne University: Mr. Morris Friedman, Mr. Joseph Levin.

Professor Basilio Manià of the University of Milan died September 26, 1939.

The death of Gaetano Bernardino Scorza, an editor of Annali di Matematica Pura ed Applicata, has been announced.

The death of Dr. H. T. Burgess has been reported. He had been a member of the Society since 1909.

Dr. S. C. Harry of Baltimore, Maryland, died September 19, 1939.

The death of Reverend Albert O'Brien has been reported. He had been a member of the Society since 1922.

Professor Emeritus F. N. Willson of Princeton University died on November 15, 1939, at the age of eighty-three years. He joined the Society in 1891.

The death of Professor Emeritus Ruth G. Wood of Smith College has been reported. She had been a member of the Society since 1899.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

138. G. E. Albert: *On contiguous point spaces and their applications.*

In topologizing the class P^* of cyclic elements of a Peano space P, R. L. Moore (Rice Institute Pamphlet, vol. 23, no. 1) was led to the concept of contiguity spaces. The presence of pairs of contiguous points seems to make the theory of Moore's spaces quite complicated. In a simpler axiomatic treatment of contiguity, Theodore Hailperin (this Bulletin, vol. 45 (1939), p. 172) modified Hausdorff's neighborhood space by denying the separation axiom and defining two points as contiguous if every neighborhood of each contains the other. The symmetry of this definition implies that the relation is transitive; this should render his theory inapplicable to the study of P^*. In the present paper a neighborhood space is introduced which differs from ordinary topological spaces in essentially one respect, namely, the Hausdorff separation axiom is replaced by the unsymmetrical Kolmogoroff axiom: of every two points, at least one has a neighborhood not containing the other. If every neighborhood of the point x contains the point $y \neq x$, then x is said to be contiguous to y. It is the purpose of the paper to construct, on this basis of contiguity, a simple and comprehensive theory of the class P^*. (Received December 11, 1939.)

139. Garrett Birkhoff: *On a class of positive matrices.*

The paper deals with matrices of non-negative elements, the sum of the terms in every row and column being the same. It is first shown combinatorially that every such matrix is a positive linear combination of permutation matrices. Applications are then made to the theory of dependent probabilities, where such matrices occur; in particular, an ergodic theorem over arbitrary semigroups is proved—valid also for an analogous class of linear (stochastic) operators on the space (L). (Received December 21, 1939.)

140. Nathaniel Coburn: *A characterization of Schouten's and Hayden's deformation methods.*

Another approach to deformation problems, in an n-dimensional space with a connection L_n, is given. The principal idea is that all deformations take place in the local tangent space E_n at each point of L_n. First, the laws of deformation of differential and ordinary vector fields are examined. These laws are distinct in non-Finsler space; they coincide in Finsler space. It is shown that: (1) Schouten's deformation formulas are obtained when the coordinate differential vectors and the unit affinor (A_μ^λ) of L_n are deformed by E_n parallelism; (2) Hayden's deformation formulas are obtained when the measure vectors of L_n are deformed by L_n parallelism, and the vectors of a subspace as well as the unit affinor of L_n are deformed by E_n parallelism. Finally, a

particular type of deformation condition is examined. The following theorems are proved: (1) if this deformation is compatible with a motion in L_n, then the Christoffel symbols of L_n are not deformed; (2) if this deformation is possible, then L_n is generally a space of absolute parallelism; (3) subcases of (2) are discussed. (Received December 4, 1939.)

141. Nathaniel Coburn: *Conformal unitary spaces.*

In this paper the properties of unitary spaces of n dimensions K_n and $'K_n$ whose fundamental tensors are related by the conformal transformations are studied. First, the relation between conformal spaces and the geodesics of these spaces is discussed. The following theorems are proved: (1) two unitary spaces, both with symmetric connection (without torsion), cannot be conformal; (2) if two unitary spaces are conformal, then their geodesics correspond; (3) conversely, if two unitary spaces, one of which is without torsion, are in a *restricted* geodesic correspondence and if a certain curvature affinor $'C_{\alpha\dot{\mu}\dot{\nu}}^{\dot{\beta}}$ vanishes, then the two spaces are conformal. The paper concludes with a study of some special conformal unitary spaces. After defining the terms: (1) unitary k-spaces; (2) k-spaces which are k-conformal, the following theorems are proved: (1) the affinor $'C_{\alpha\dot{\mu}\dot{\nu}}^{\dot{\beta}}$ is a conformal invariant of all k-spaces which are k-conformal; (2) conversely, if two unitary spaces are conformal and the affinor $'C_{\alpha\dot{\mu}\dot{\nu}}^{\dot{\beta}}$ is a conformal invariant, then the spaces are k-spaces. A special case of these k-spaces has been studied by Bergmann, Mitrochin, Fuchs and others. (Received December 4, 1939.)

142. W. C. Graustein: *Harmonic minimal surfaces.*

A minimal surface in a euclidean space of three dimensions is harmonic if it is representable in terms of Cartesian coordinates (x_1, x_2, x_3) by an equation of the form $U(x_1, x_2, x_3)=\text{const.}$, where U is a harmonic function. It is the purpose of this paper to determine all the families of harmonic minimal surfaces. The families found are a pencil of planes, a family of right helicoids, a family of imaginary transcendental surfaces, a family of imaginary quartic surfaces, a family of imaginary sextic surfaces (for which U depends on an elliptic integral), and families of imaginary cylinders with isotropic rulings. The method employed introduces three mutually orthogonal congruences of curves, with unit tangent vector fields α, β, γ, which are closely associated with the required family of surfaces, and expresses the prescribed properties of these surfaces by a suitable choice of the coefficients in the equations of variation of α, β, γ, with respect to the arcs of the curves of the three congruences. These equations of variation constitute the differential system finally to be integrated. Their conditions of integrability yield a second differential system of ten partial differential equations of the first order in five dependent and three independent variables. (Received December 7, 1939.)

143. D. W. Hall and G. T. Whyburn: *On arc-preserving and tree preserving transformations.*

In an earlier abstract (45-11-399) the authors have obtained results concerning arc-preserving transformations. In the present paper the following additions to the theory are obtained: (a) Condition (i) (a) of the earlier abstract characterizes tree preserving transformations. (b) If $T(A)=B$ be continuous, then in order that T be tree preserving it is necessary and sufficient that the image of every simple arc in A be a tree in B. (c) If B is cyclic, then the following types of transformations are equivalent: arc-preserving, tree preserving, A-set reversing, monotone retracting. The

set A is, throughout the paper, a compact locally connected continuum. (Received December 27, 1939.)

144. P. R. Halmos: *Statistics, set functions, and spectra.*

The purpose of this paper is to exploit the analogy existing between the distribution functions of statistics on the one hand and spectral families of projection operators on the other hand. It is shown that the class of all bounded measurable functions in the unit interval satisfies the axiomatic system developed by von Neumann (Matematicheskii Sbornik, vol. 43 (1936)) for the class of all bounded Hermitian operators on Hilbert space. Certain well known theorems, such as the equality of the expectation of a chance variable with the first moment of its distribution function, and the Radon-Nikodym theorem concerning absolutely continuous set functions are special cases of the general theory considered. These theorems are suggested by the application of Hilbert space theorems to real function theory. Working in the converse direction, certain concepts (particularly independence) of real function theory are extended to operators and the behavior of sets of independent operators is investigated, yielding results parallel to the usual results of probability theory. (Received January 3, 1940.)

145. E. D. Hellinger: *Classes of monotone functions.* Preliminary report.

The problem is to investigate the manner of increase of monotone functions over an interval or over a perfect set of points and to form classes of functions with common behavior in this respect. A function $g(x)$ is in the class defined by $h(x)$ if it can be represented by an integral of the form $\int_a^x (df(x))^2/dh(x)$ with a suitable $f(x)$; the integral is of the kind introduced by the author as a generalization of the Stieltjes integral (T. H. Hildebrandt, this Bulletin, vol. 24 (1918), p. 194). By these and other related integrals conditions are found under which two functions shall belong to the same class; further representatives of the different classes are given. It is shown that the theory is closely connected with the theory of absolute continuity. The theory can be applied, for instance, to the problem of the orthogonal equivalence of infinite quadratic forms or symmetric operators, bounded as well as non-bounded, which is just the problem which suggested these investigations. (Received December 27, 1939.)

146. Henry Hurwitz: *Total regularity of infinite matrix transformations.*

Total regularity of a matrix transformation was defined by W. A. Hurwitz who found a necessary and sufficient condition for it in the case of a real infinite triangular matrix (this Bulletin, vol. 28 (1922), p. 30). In the present paper a necessary and sufficient condition is deduced for a real regular infinite square matrix transformation given by $X = \lim \sum_{k=0}^{\infty} a_k(t) x_k$. There must eventually be only a finite number of negative elements $a_k(t)$ for each t, and there must be no sequence t_i such that a suitably defined set of negative elements appearing in the sequence is not properly "guarded" by large positive elements which also appear. Several modifications of the condition are possible. (Received December 15, 1939.)

147. A. N. Lowan and Gertrude Blanch: *Analysis of computing errors in the process of analytic continuation.*

When the values of a function $f(x)$ are to be tabulated over a fairly wide range of the argument x, it is frequently convenient to compute the function and its deriva-

tives for certain equally spaced "key arguments" and to sub-tabulate by means of Taylor expansions around the key arguments. (It is assumed that all required derivatives are continuous over the entire range.) The scheme adopted by the "Project for the Computation of Mathematical Tables," W.P.A., New York City, involves computing in succession derivatives of all orders needed for every point in the range, beginning with those of the highest order, and ending with the function itself, keeping the interval constant. No estimate of the manner in which the error in the method is propagated from stage to stage seems to be available in the literature, and because of the importance of the method as a means of computation, an upper bound of the error is derived in this paper. (Received December 20, 1939.)

148. A. V. Martin and J. H. Roberts: *Two-to-one transformations of 2-manifolds.* Preliminary report.

If A is a compact 2-manifold, then a continuous two-to-one transformation can be defined on A if and only if the Euler characteristic $\lambda(A)$ is even. If B is the image space, then $\lambda(B)=\lambda(A)/2$. The possible image spaces include all the manifolds B (in number 0, 1, or 2) with $\lambda(B)=\lambda(A)/2$, as well as certain 2-complexes obtained by identifying pairs of points of a manifold. (Received December 29, 1939.)

149. Marston Morse and C. B. Tompkins: *Minimal surfaces of unstable type.*

The authors continue the theory initiated in the April issue of the Annals of Mathematics extending it to surfaces bounded by several contours. A typical result extending a classical result well known for surfaces of revolution bounded by two circles and generated by a catenary is as follows. Let g_1 and g_2 be two simple rectifiable closed curves whose convex envelopes do not intersect. Suppose that the ratio of chord to arc on g_i, $(i=1, 2)$, is bounded from zero. Term a surface S a ring-surface or disc-surface if S is the continuous image of a circular ring or a circular disc, respectively. Referring the term "minimizing" to the Douglas-Dirichlet integral, our theorem states the following. If g_1 and g_2 bound a ring mimimal surface of minimizing type (relative or absolute), then g_1 and g_2 also bound a ring minimal surface not of minimizing type, or else at least one of the two contours g_i bounds a disc minimal surface of unstable (non-minimizing) type. This result is easily verified for the classical case of surfaces of revolution, where the ring surface of unstable type always exists under the hypotheses. (Received December 1, 1939.)

150. Tibor Radó: *On a lemma of McShane.*

In his work on the semi-continuity of double integrals in the calculus of variations, McShane (Annals of Mathematics, (2), vol. 34 (1933), pp. 829–830, Lemma 7) established an important lemma which plays a fundamental part in his proofs. Using the notations of that paper, the main result of the present paper can be described by the following statement. The conclusion of the lemma of McShane remains valid if the functions x_0^i are subjected to the (obviously necessary) condition that the Jacobians X_0^i be summable. The proof is based on methods developed by the author in his work on the area of surfaces. Applications will be considered on another occasion. (Received December 13, 1939.)

151. A. L. Whiteman: *Additive prime number theory in real quadratic fields.*

In a series of three papers (Abhandlungen aus dem Mathematischen Seminar der

Hamburgischen Universität, vol. 3 (1924), pp. 109–163 and pp. 331–378; Mathematische Zeitschrift, vol. 27 (1926), pp. 321–426) Rademacher considered the problem of representing numbers in an algebraic field as the sum of primes. By applying the Hardy-Littlewood method he showed that if a certain hypothesis concerning the distribution of the zeros of Hecke's $\zeta(s, \lambda)$-functions is true, then every sufficiently "large" "odd" number in an algebraic field is the sum of three primes. In this paper it is proved that the same hypothesis implies that "almost" every "even" integer in a real quadratic field is the sum of two primes. The proof makes use of the Hardy-Littlewood-Rademacher method and also employs certain simplifications due to Landau. (Received December 22, 1939.)

152. R. P. Agnew: *Tauberian conditions.*

Let $K>0$ and $0<\theta<\pi/2$. Corresponding to each angle ψ, let $S(K, \theta, \psi)$ denote the "sector with vertical angle less than π" consisting of all points z of the complex plane representable in the form $z=-Ke^{i\psi}+\rho e^{i(\psi+\phi)}$ where $\rho \geqq 0$, $-\theta \leqq \phi \leqq \theta$. A series $\sum u_n$ of complex terms is said to satisfy the Tauberian condition T if K, θ, λ, and $\psi_1, \psi_2, \psi_3, \cdots$ exist such that $K>0$, $0<\theta<\pi/2$, $\lambda>1$, and, for each sufficiently great index k, $nu_n \in S(K, \theta, \psi_k)$ when $k \leqq n<\lambda k$. The condition T is sufficiently general to include classic Tauberian conditions of "order" and "gap" types, and hence provides the means of both generalizing and unifying classic Tauberian theorems. Characterizations and properties of the class of series satisfying T are developed. A condition T^* which is more general than T (and is related to T in much the same way that the condition that $\sum u_n$ be "langsam oszillierende" is related to the condition $n|u_n|<K$) is given, characterized, and discussed. Tauberian theorems involving T and T^* are proved for C_1 and other methods of summability. (Received January 10, 1940.)

153. R. P. Agnew and Mark Kac: *Translated functions and statistical independence.*

In connection with some problems of Kampé de Feriet concerning the theory of turbulence, Steinhaus proposed the following question: Does there exist a continuous function $f(t)$, $(-\infty<t<\infty)$, such that, for each sequence of real numbers $\lambda_1, \lambda_2, \cdots$, the functions $f(t+\lambda_1), f(t+\lambda_2), \cdots$ are statistically independent? (For the definition of statistical independence see M. Kac and H. Steinhaus: *Sur les fonctions indépendantes* IV, Studia Mathematica, vol. 7, pp. 1–15, and P. Hartman, E. R. van Kampen and A. Wintner: *Asymptotic distributions and statistical independence*, American Journal of Mathematics, vol. 61 (1939), pp. 477–487.) A positive answer to this question is given by the example $f(t)=\sin e^{t^2}$. A more general class of functions having the above property can be obtained. It may be mentioned that once the statistical independence is established, one can get different results concerning superposition of translations of the function. For instance, the relative measure of the set of those t's for which $f(t+\lambda_1)+\cdots+f(t+\lambda_n)<\alpha(n)^{1/2}$, $(f(t)=\sin e^{t^2})$, tends to $\pi^{-1/2}\int_{-\infty}^{\alpha} \exp(-u^2)du$ as $n\to\infty$. (Received January 13, 1940.)

154. G. E. Albert: *On quasi-metric spaces.* Preliminary report.

The present note studies a modified form of the quasi-metric spaces introduced by W. A. Wilson (American Journal of Mathematics, vol. 53 (1931), p. 675). Here, a space S will be called quasi-metric if for every pair of points x, y in S two non-negative numbers xy and yx can be defined such that (1) $xy=0=yx$ if and only if $x=y$, and (2) $xz \leqq xy+yz$ for every triple of points x, y, z in S. (Wilson used the axiom: $xy=0$ if and only if $x=y$.) If $xy=0$ but $yx\neq 0$, then x is called contiguous to y. Let $\{x_n\}$ be

a sequence of points; one says that $\{x_n\}$ converges to x as an l-limit, r-limit, or c-limit accordingly as $xx_n \to 0$, $x_n x \to 0$, or both, respectively. One defines the corresponding classes of l-closed, r-closed, and c-closed sets as well as l-, r-, and c-open sets. The relationships between quasi-metric spaces and topological spaces are studied in detail. In particular, it is shown that the former include as special cases the contiguity spaces recently defined by the author (abstract 46-3-138) with neighborhood as the primitive concept. (Received January 30, 1940.)

155. Reinhold Baer: *Abelian groups that are direct summands of every containing abelian group.*

A characterization of those abelian groups, admitting a ring of operators, which are direct summands of every containing abelian group is given. It is shown that they are essentially a generalization of the groups "with division" and that they have similar properties. (Received January 5, 1940.)

156. A. C. Berry: *The accuracy of the Gaussian approximation to a sum of independent variates.*

The sum of finitely many variates frequently possesses an almost Gaussian distribution. Let M denote the least upper bound of the modulus of the difference between the distribution function of the variate sum and that of a related normal variate. M measures the "abnormality" of the variate sum. For uniformly bounded, totally independent variates, the following inequality is established: $M \leqq 2L/\sigma$. Here L is the least common bound of the *individual* variates, none of the given variates differs from its mean value by more than L except at most in cases of zero probability, and σ is the standard deviation of the variate *sum*. This result is an arithmetical refinement of the theorem of Liapounoff that $M \to 0$ when $L/\sigma \to 0$. The constant 2 is not the best possible. The existence of a best constant is proved, also that it cannot be less than $(2\pi)^{-1/2}$. The result rests on an equality connecting the difference of distribution functions with the difference of *characteristic* functions, an instance of the Parseval theorem in the theory of Fourier transforms. With the aid of a method of Paul Lévy, an extension is obtained for the case of unbounded variates. (Received January 25, 1940.)

157. W. Z. Birnbaum and H. S. Zuckerman: *On the properties of a collective.*

This paper is concerned with the concepts of a "collective" and a "selection" ("Stellenauswahl"), used in von Mises' foundations of the calculus of probabilities. The central theorem is: The set of all infinite selections can be interpreted as a space in which a Lebesgue measure is defined; then, if a sequence of 0's and 1's fulfills the first postulate of von Mises, it fulfills the second postulate in the sense that the limit of the relative frequencies is invariant for almost all selections in the space. (Received January 22, 1940.)

158. A. B. Brown: *On transformation of multiple integrals.*

The formula for transformation of a multiple Riemann integral under a change of variables $x_i = x_i(u_1, \cdots, u_n)$, $(i = 1, \cdots, n)$, depends on the formula for the volume V in (x)-space of the image, under the transformation, of a solid R in (u)-space, namely $V = \int \cdots \int_R |J_{xu}| du_1 du_2 \cdots du_n$, where J_{xu} is the Jacobian. The standard proof of the latter formula is sufficiently forbidding, for $n \geqq 3$, to discourage one's

presenting it to a class in advanced calculus. In the case $n=2$, the proof does not make it clear what formula will be obtained when n is greater than 2. A different method of proof gives the result quickly and naturally, but has not been given completely in the literature, to the writer's knowledge. In this note a simple but complete derivation of the formula is given following the second method. (Received January 17, 1940)

159. L. E. Bush: *An asymptotic formula for the average sum of the digits of integers.*

Let $S(r, N)$ be the sum of the digits of all non-negative integers less than N, when these numbers are written in the scale of notation of radix r, so that $S(r, N)/N$ is the average sum of the digits of all numbers less than N (including zero) when these numbers are written in the r-scale. An asymptotic formula is found for $S(r, N)/N$ by which it is shown that for a sufficiently large N the average sum of the digits of all non-negative integers less than N is least when the numbers are written in the binary scale, and in general is smaller for a smaller radix than for a larger. (Received January 19, 1940.)

160. J. Hobart Bushey: *Expansions of products of certain symmetric orthogonal polynomials.*

In 1878 Adams gave (Proceedings of the Royal Society of London, vol. 27, pp. 63–71) a formula for the expansion of the product $P_m(x)P_n(x)$ of two Legendre polynomials in a series of Legendre polynomials, and made use of this expansion to evaluate $\int_{-1}^{1}P_m(x)P_n(x)P_r(x)dx$, $(m, n, r=0, 1, 2, \cdots)$. In the present paper, analogous results are obtained for symmetric Jacobi polynomials and for Hermite polynomials. (Received January 29, 1940.)

161. Sister M. Patricia Callaghan: *Generalized Frégier curves.*

The author finds equations of envelopes of generalized Frégier theorems. If in the Frégier theorem a constant angle is used instead of the right angle, the envelope is a conic. The equation of the envelope of this envelope is found as the point moves around the given conic. If through any fixed point on a conic, pairs of lines are drawn making supplementary angles with a fixed line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola. The equation of the envelope of this envelope is found as the fixed point moves around the given conic. (Received January 27, 1940.)

162. R. H. Cameron and W. T. Martin: *An unsymmetric Fubini theorem.*

Under fairly general conditions the authors prove that $\int s(u)d\int p(x, u)dk(x) = \int dk(x)\int s(u)dp(x, u)$, where the integrals are Lebesgue-Stieltjes (Radon) integrals and all the limits are from $-\infty$ to ∞. (Received January 12, 1940.)

163. Richard Courant and N. Davids: *Minimal surfaces spanning closed manifolds.*

The methods for the solution of the Plateau-Douglas problem must be considerably modified if "free boundaries" on prescribed boundary manifolds are considered. The present paper, extending previous results (Proceedings of the National Academy of Sciences, vol. 24 (1938), p. 97 ff., and Acta Mathematica, vol. 72 (1940)) solves the problem of constructing an extremal minimal surface whose boundary is free on

a given closed manifold of genus $p>0$. The boundary is required to be linked with a prescribed cycle which itself is linked with the manifold. This novel type of topological conditions in the calculus of variations is essential for the formulation and solution of a wide class of problems concerning minimal surfaces. (Received January 17, 1940.)

164. M. M. Day: *Linear methods of summability.*

Let Y be a directed set; that is, a partially ordered set with the transitivity and composition properties of Moore and Smith (American Journal of Mathematics, vol. 44 (1922), pp. 102–121). Let V be a space of real-valued functions on Y such that $\lim_y f(y)$ exists in the Moore-Smith sense if $f \in V$. Let X be another directed set and for each $x \in X$ let U_x be a functional on V; the transformation thus defined is regular on V if $\lim_x U_x(f) = \lim_y f(y)$ for all $f \in V$. Extensions of the theorem of Toeplitz for simple sequences are shown to hold when V is a Banach space and the set X is essentially sequential in the sense that there exists a sequence $\{x_n\} \subset X$ such that each $x \in X$ is followed by some x_n. The weak topologies in the conjugate space of V and the descriptive theory of directed sets, due to Tukey (Thesis, Princeton University, 1939), are used in showing that some such restriction is necessary. The results are applied to finding regularity conditions for various function classes defined on particular directed sets, such as multiple sequences, functions of n real variables, continuous functions, and unconditionally convergent series. (Received January 26, 1940.)

165. M. M. Day: *The spaces L^p with $0<p<1$.*

The class $L_\mu^p(Y)$ is the set of all real-valued functions f defined on Y, measurable μ and such that $\int_Y |f|^p d\mu < \infty$, where μ is a measure (Saks, *Theory of the Integral*, chap. 1) and $0<p<1$. It is shown that a non-identically-zero, additive, and continuous functional on $L_\mu^p(Y)$ exists if and only if there is a measurable set E with $0<\mu(E)<\infty$, such that E cannot be divided into two disjoint sets of positive measure. The proof proceeds stepwise, first assuming $\mu(Y)<\infty$, next that Y can be split into a countable sum of sets of finite measure, and finally removing all such restrictions. As a corollary, no such nonzero functionals exist when Y is any measurable subset of a euclidean n-space and μ is Lebesgue measure. In the case when they exist, a general form of the linear functionals is given, and the class of them is shown to be equivalent to the class of bounded functions on a properly chosen set. (Received January 22, 1940.)

166. R. P. Dilworth: *A characterization of lattices of ideals.*

An element a of a continuous lattice is said to be finite dependent if $a \supset \Pi(S)$ implies $a \supset \Pi(S')$ where S' is a finite subset of S. The following theorem is proved: A continuous lattice $\mathfrak{S}$ is isomorphic to the lattice of ideals of a sublattice if and only if the finite dependent elements are closed with respect to union and generate $\mathfrak{S}$ under infinite crosscut. It follows that every exchange lattice is a lattice of dual ideals. (Received January 29, 1940.)

167. F. G. Dressel: *A Stieltjes integral equation.*

The paper shows how to solve a particular type of Young-Stieltjes integral equation. Also it is shown that only one of the three conditions imposed by Fischer (Annals of Mathematics, (2), vol. 25 (1923–1924), pp. 142–158) on the kernel of a Stieltjes integral equation is needed to insure the solution of this equation. (Received January 24, 1940.)

168. F. G. Dressel: *The fundamental solution of the parabolic equation.*

The existence of the fundamental solution for the parabolic equation with variable coefficients is proved in this paper. (Received January 24, 1940.)

169. W. K. Feller: *On the integrodifferential equations of the purely discontinuous Markoff processes.*

Consider a random variable $X(t)$ varying in some space E so that if $X(t)$ coincides with the point x the probability that no change will occur during $(t, t+\Delta t)$ is $p(t, x)\cdot\Delta t+o(\Delta t)$, and the probability of $X(t+\Delta t)$ being contained in the set $\Lambda \ni x$ is $p(t, x)\Pi(t, x, \Lambda)\cdot\Delta t+o(\Delta t)$, where $\Pi(t, x, \Lambda)$ is a probability distribution. It is examined to what extent these conditions determine the transition probabilities $P(\tau, x; t, \Lambda)$ of some stochastic process. Generalizing previous results (*Zur Theorie der stochastischen Prozesse*, Mathematische Annalen, vol. 113 (1936), and W. Dubrovski, *Eine Verallgemeinerung der Theorie der rein unstetigen stochastischen Prozesse von W. Feller*, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 19 (1938)) it is shown that, if all functions depend continuously on t, a derivative $\partial P/\partial t$ exists for a suitable class of sets and almost all t. The problem reduces to two integrodifferential equations which determine uniquely an additive function of sets $P(\tau, x; t, \Lambda)$ with $0\leqq P\leqq 1$. In the case of an unbounded $p(t, x)$, however, $P(\tau, x; t, E)$ may fall short of unity, although all other requirements of the theory are always fulfilled. In the case of temporally homogeneous processes, necessary and sufficient conditions for $P(\tau, x; t, E)=1$ are given, which are related to the ergodic properties of the system. (Received January 28, 1940.)

170. Tomlinson Fort: *Summability and the definition of a limit.*

Let $S_0^x f(t)\Delta_t$ denote a principal solution of the difference equation $\Delta_x y=f(x)$ as defined by Nörlund. In the present paper a study is made of the transformation $S_0^x K(x, t)f(t)\Delta_t$ analogous to studies that have been made of the integral transformation $\int_0^\infty K(x, t)f(t)dt$. A variety of sufficient conditions are obtained that the transformation be regular, limit-producing, and so on. (Received January 18, 1940.)

171. H. L. Garabedian and H. S. Wall: *Hausdorff matrices and continued fractions.*

In a recent paper (abstract 46-1-130) Wall characterized totally monotone sequences in terms of continued fractions. In this paper the subclass of regular totally monotone sequences is so characterized. If $c_0, c_1, c_2, \cdots$ is totally monotone, then the row, column, and diagonal sequences of the difference matrix $(\Delta^m c_n)$ are totally monotone. Necessary and sufficient conditions are obtained in terms of the continued fraction in order that these be regular sequences. This is accomplished by means of the curious result that when $c_0-c_1x+c_2x^2-\cdots\sim c_0/1+g_1x/1+g_2(1-g_1)x/1+g_3(1-g_2)x/1+\cdots$, then $c_0-\Delta c_0x+\Delta^2c_0x^2-\Delta^3c_0x^3\cdots\sim c_0/1+(1-g_1)x/1+g_1g_2x/1+\cdots$, where the second continued fraction is obtained from the first by replacing g_{2n-1} by $1-g_{2n-1}$, $(n=1, 2, 3, \cdots)$. A hypergeometric summability obtained from the continued fraction of Gauss is investigated as to inclusion relations, and the methods of summability arising from the difference matrix for the base sequence are considered. Finally a general theorem is given on the effectiveness of the methods considered relative to analytic continuation of power series outside the circle of convergence. (Received January 13, 1940.)

172. G. A. Hedlund: *A new proof for a metrically transitive system.*

Two distinct methods have been used to prove that the flows defined by the geodesics on suitably restricted surfaces of constant negative curvature are metrically transitive. The first of these (cf. Annals of Mathematics, (2), vol. 35 (1934), p. 787) involves the use of symbolism to characterize the geodesics and is restricted to those surfaces for which a suitable symbolism has been devised. The second of these methods (cf. E. Hopf, *Ergodentheorie*) makes use of the theory of harmonic functions and is valid for all complete surfaces of constant negative curvature and of finite area. Both of the methods seem to involve excessive machinery and it would seem desirable to have a simpler and more straightforward proof of this result. The present paper gives a new and elementary proof of the metric transitivity of the flow defined by the geodesics on any closed orientable surface of constant negative curvature. The method can be readily extended to the case of complete surfaces of constant negative curvature and finite area. (Received January 25, 1940.)

173. G.A. Hedlund: *Surfaces of negative curvature and metric transitivity.*

Let U be the unit circle $z\bar{z}=1$, $z=x+iy$, and let Ψ be its interior. The author considers the quadratic form (A) $ds^2=\lambda^2(x, y)(dx^2+dy^2)/(1-x^2-y^2)^2$, $(x^2+y^2<1)$; where $\lambda(x, y)$ is a function defined in Ψ and satisfying the following conditions: (I) $\lambda(x, y)$ is of class C^2 in Ψ; (II) there exist positive constants a and b such that $a\leqq\lambda(x, y)\leqq b$ in Ψ; (III) there exist positive constants c and d such that if $K(x, y)$ denotes the Gaussian curvature of (A), $-d^2\leqq K(x, y)\leqq -c^2$ in Ψ; (IV) there exists a positive constant e such that if $H(P, Q)$ denotes the hyperbolic distance between arbitrary points P and Q of Ψ, then $|K(P)-K(Q)|\leqq eH(P, Q)$; (V) $\lambda(x, y)$ is invariant under a Fuchsian group which has U as principal circle, which is of the first kind, and which has a finite number of generators. If points which are congruent under F are considered identical, there is defined a manifold M of negative curvature. It is shown that the geodesic system on M is metrically transitive. The result has been attained previously only in the case when $K(x, y)$ is a negative constant. (Received January 25, 1940.)

174. Fritz Herzog: *Uniqueness theorems for rational functions.*

The uniqueness theorems, dealt with in this paper, are of the type of Nevanlinna's theorem which says that two meromorphic functions must be identical if they have *five* identical distributions, that is, if they assume five complex values (finite or infinite) at the same points. No assumption whatever is made about the multiplicity with which the two functions assume these five values at the various points. While five is the smallest number for which this uniqueness theorem holds true for meromorphic functions, the paper shows that for rational functions, *four* identical distributions are sufficient for the identity of the two functions. *Three* identical distributions, however, are not sufficient to insure the identity of the two functions. Several examples of pairs of rational functions with three identical distributions are given, also a triplet of such functions. Under certain restrictive conditions, concerning for instance the degrees of the two functions, uniqueness theorems for rational functions with three identical distributions are derived. It is also shown that two polynomials with identical distributions with respect to two finite values are identical. Finally, there can be no more than a finite number of rational functions with given distributions with respect to three given values. (Received January 6, 1940.)

175. F. B. Hildebrand and P. D. Crout: *A least square procedure for improving a method for solving integral equations by polynomial approximation.*

This paper continues the development of an approximate method for solving integral equations begun in a recent paper (Crout, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 19 (1940), pp. 34–92). This method has given very good results when applied to a number of physical problems, and consists essentially in replacing the unknown function by an algebraic polynomial determined by $2n+1$ unknown ordinates, using for convenience the Lagrangean form, tables for which are given. By interchanging the order of integration and summation and requiring that the integral equation be satisfied at a number of points, there is obtained a set of linear equations in the unknown ordinates which fix the polynomial. The least square process described is written into a single matrix multiplication, and gives a great increase in accuracy compared with the original technique; furthermore it permits a simple consideration of the error. In setting up physical problems and in applying the method described for solving the resulting integral equations, the work is greatly simplified by the continual appearance of matrices; also the work involved in any one problem automatically includes most of that required by a variety of other problems. (Received January 30, 1940.)

176. Einar Hille and I. E. Segal: *Blaschke products as Laplace-Stieltjes integrals.*

The possibility of representing a Blaschke product $B(z)$ of the half plane type as an absolutely convergent Laplace-Stieltjes integral in some half plane is investigated. It is shown that such a representation is impossible if the zeros of $B(z)$ are all real and tend to infinity, or if the zeros satisfy certain other conditions. A necessary and sufficient condition for $(z+1)^{-\alpha}B(z)$, $(0<\alpha\leqq 1/2)$, to be representable as an absolutely convergent Laplace-Stieltjes integral is given for the case where the zeros are all real and tend to infinity. The result is qualitatively that the more slowly the zeros tend to infinity, the larger α must be in order for $(z+1)^{-\alpha}B(z)$ to be representable as an absolutely convergent Laplace-Stieltjes integral in some half plane. (Received January 26, 1940.)

177. S. A. Jennings: *Nilpotent groups and nilpotent rings.*

Consider a nilpotent ring N which is free of operators and which satisfies an ascending chain condition for two sided ideals. Such a ring may be generated by a finite number of elements, and possesses a finite basis $n_1, \cdots, n_r$ so that any $n \in N$ may be written $n=\sum\alpha_i n_i$, where $\alpha_i n_i$ denotes $n_i+n_i+\cdots+n_i$ to α_i terms. Adjoin a unit element 1 to N and consider the set of elements of the form $1+n$. These elements form under multiplication a nilpotent group G, which is generated by a finite number of elements and whose class is not greater than the exponent of N. The order of G will be finite or infinite as the order of the additive group of N is finite or infinite. Conversely, let G be any nilpotent group with a finite number of generators. By using a result of W. Magnus (Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 105–115) on the representation of the free group in a certain ring, it is shown that G determines a nilpotent ring of the above type, whose exponent is not greater than the class of G. There is, therefore, a 1:1 correspondence between nilpotent rings with a finite basis and nilpotent groups generated by a finite number of elements. (Received January 30, 1940.)

178. I. N. Kagno: *On a certain non-separating graph on an orientable surface.*

Let G be a connected linear graph, containing no vertices of degree less than 3, which is mapped on an orientable surface Σ_p of genus $p>0$ in such a manner that Σ_p-G is a single 2-cell. The structure of G is determined and it is shown that G has at most $6p-3$ arcs, and Σ_p-G has at most $12p-6$ sides. This result is applied to the universal covering surface of Σ_p and it is shown that a fundamental domain of the group of *decktransformations* of the covering surface has at most $12p-6$ sides. In particular, if Σ_p is an algebraic Riemann surface of genus $p>1$, the results of this paper can be applied to determine an upper bound for the number of sides of a fundamental domain of the group of linear transformations to which an automorphic function belongs. (Received January 19, 1940.)

179. E. R. Kolchin: *On the exponents of differential ideals.*

After defining the concept, "exponent of a differential ideal," theorems are given connecting the exponent of a differential ideal with the exponents of the factor ideals in the representation given by J. F. Ritt (Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 90–91) and showing the invariance of the exponent under differential field adjunction. These results are applied to the study of the exponent of the differential ideal generated by a single form in one unknown of order unity. It is seen that much depends on the nature of the singular solutions of the form. Complete results are not obtained, but the exponent is determined for a wide class of such ideals. Finally, there is a brief discussion of chains of differential ideals. A theorem is given relating the length to the exponent of a principal chain. (Received February 3, 1940.)

180. D. P. Ling and Leon Recht: *Geodesics on a paraboloid of revolution.* Preliminary report.

This paper is concerned with the study of the system of geodesics on a paraboloid of revolution. It is shown that the surface is divided into an infinite number of "zones" bounded by circles whose planes are perpendicular to the axis. The zone containing the vertex is characterized by the fact that no geodesic starting from a point in this zone can return to its starting point. Through every point in the next zone there is exactly one geodesic which returns to this point after one turn around the paraboloid, but none which returns after more than one turn. Through a point in the kth zone there are k geodesics which return after $1, 2, \cdots, k$ turns respectively, but none which returns after more than k turns. Finally, a method for approximating the positions of the circles between these zones is given. (Received January 19, 1940.)

181. J. K. L. MacDonald: *Existence theorems for node and amplitude properties near singularities of linear differential equations.*

Necessary and sufficient conditions for forms of solutions of linear, ordinary, second order differential equations near singular end points are established. Comparison, transformation and iteration theorems are generalized on the basis of previous work of the writer (this Bulletin, vol. 45 (1939), pp. 164–171, and abstract 44-9-356). Comparison criteria (analogous to the sequence of logarithmic tests for series convergence and divergence) are developed for determining whether or not nodes accumulate near the singular end points. Transformations leading in all cases to coefficients of the same sign are discussed. The phase angle and the transformed linear equations are expressed in forms which show their interrelationship in a clear way. (Received January 27, 1940.)

182. Saunders MacLane: *Note on the relative structure of p-adic fields.*

A problem of relative structure for p-adic fields, which arose in a previous paper (Annals of Mathematics, (2), vol. 40 (1939), pp. 423–442), can now be solved. Let k be a p-adic field with a residue class field $\mathfrak{k}$ of characteristic p. Let $\mathfrak{K}$ be any extension of this residue class field. A necessary and sufficient condition that any two p-adic extensions K and K' of k, both having the same residue class field $\mathfrak{K}$, be analytically equivalent over k is the requirement that the given residue class extension $\mathfrak{K}/\mathfrak{k}$ preserve p-independence (see definition in Duke Mathematical Journal, vol. 5 (1939), pp. 372–393). Here an *analytical equivalence* of K to K' is to be an isomorphism of K to K' which leaves values and residue classes of all elements in K fixed. The necessity proof is an extension of previous examples, and is based on a suitable analysis of those extensions $\mathfrak{K}/\mathfrak{k}$ which do not preserve p-independence. (Received January 29, 1940.)

183. R. S. Phillips: *On the space of completely additive set functions.*

This paper is devoted to an investigation of a class of Banach spaces typified by the space of completely additive set functions on a given Borel field of sets. The space of functions of bounded variation is a particular instance of such a space. The central result is that a space of this type can be represented as the direct product of spaces of completely additive set functions, each absolutely continuous with respect to a unique measure function of a certain orthogonal set. The author shows that to each set function $x(\tau)$ of the space corresponds a denumerable number of disjoint sets τ_i such that $x(\tau)$ is uniquely representable in the form $x(\tau)=\sum x(\tau\cdot\tau_i)$ where $x(\tau\cdot\tau_i)$ belongs to one of the component spaces. By means of this decomposition it is easy to demonstrate that the space is weakly complete and that it possesses a generalized base. Finally the author is able to define the general linear functional on the space and characterize the conjugate space. (Received January 9, 1940.)

184. Tibor Radó and P. V. Reichelderfer: *Note on an inequality of Steiner.*

Let $L(f)$ denote the Lebesgue area of a continuous surface $z=f(x, y)$ defined over the unit square Q. If $z=f_n(x, y)$ is a sequence of continuous surfaces defined over Q and converging uniformly to $z=f(x, y)$ on Q for which $L(f_n)$ and $L(f)$ are finite, then McShane (Annals of Mathematics, (2), vol. 33 (1932), pp. 125–138) has shown that smallness of $\epsilon_n \equiv [L(f_n)+L(f)]/2-L([f_n+f]/2)$ implies smallness with respect to measure of $\delta_n(x, y) \equiv [(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$, whence he concludes that if $L(f_n)$ converges to $L(f)$, then $\delta_n(x, y)$ converges to zero in measure. This result is improved in the present paper by replacing smallness with respect to measure by smallness with respect to exponent. It is shown that smallness of ϵ_n implies smallness of $I_n(\lambda) \equiv \iint_Q \delta_n^\lambda(x, y)dxdy$ for every exponent λ satisfying $0<\lambda<1$, and thus if $L(f_n)$ converges to $L(f)$ then $I_n(\lambda)$ converges to zero for every λ satisfying $0<\lambda<1$. Since McShane has shown that $I_n(1)$ does not generally converge to zero when $L(f_n)$ converges to $L(f)$, the present result is the best obtainable under the stated hypotheses. (Received January 19, 1940.)

185. Eric Reissner: *On Green's function of $\nabla^2\nabla^2 w=0$ for the half plane.*

The following result, of interest mainly for the theory of bending of thin elastic plates, is proved. The explicit form of Green's function of $\nabla^2\nabla^2 w=0$ for the half

plane $x \geq 0$ satisfying the boundary conditions $w=0$ and $(\partial w/\partial x)=\lambda \nabla^2 w$ along $x=0$ is: $w=(1/8\pi)\Re n\left[|z-a|^2 \log (z-a)/(z+a)+2az+8ax \exp((z+a)/2\lambda)\,\mathrm{Ei}((z+a)/2\lambda)\right]$. In the formula for w, Ei (x) denotes the "exponential integral." The solution for the special case $\lambda=0$ has been given by J. H. Michell (Proceedings of the London Mathematical Society, vol. 34 (1902), pp. 223–228). (Received January 25, 1940.)

186. H. J. Riblet: *Certain theorems for algebraic differential fields.*

In this paper several theorems are proved for finite algebraic differential fields. One states that in a finite algebraic differential field $\overline{K}(\theta_1)$ there exists an element which is linearly independent of its $n-1$ conjugates. This is helpful in proving that if every element of $\overline{K}(\rho)$ satisfies a linear homogeneous differential equation of order not greater than n over $\overline{K}$, then ρ is algebraic over $\overline{K}$ of degree not greater than n. (Received January 16, 1940.)

187. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* Preliminary report.

The differential equation satisfied by the generating functions $U(x, t)=\sum_{m=0}^{\infty} R_m(x) t^{m+\alpha}$ of a set of orthogonal polynomials is found to be $\sum_{i=0}^{n} t^i [a_i+(b_i x+c_i)t+d_i t^2]\cdot \partial^i U/\partial t^i=0$, where the various a's, b's, c's, and d's are constants. Depending on the values of these constants, the solution of the equation for $n-1$ gives three new functions in addition to the generating functions of the Legendre, Laguerre, and Hermite polynomials. The properties of the orthogonal polynomials generated by the new functions are investigated. The range of orthogonality and the weight factors are derived by a simple method in two of the cases. The weight factors are of the form of gamma- and beta-functions. Possible generalizations of the results are discussed. (Received January 27, 1940.)

188. Peter Scherk: *Estimations with integers in the demesne of the α-β-hypothesis.*

The "number function" $D(x)$ of a set D of positive integers d denotes the number of the d not greater than x. Let A, B be sets of positive integers respectively a, b; and let C be the set of all the numbers of the form a, b or $a+b$. The author gives two new estimates of the number function of the set C through those of A and B. One of them is the simplest possible symmetric in A and B, the other being related to it by Khintchine's "Umkehrformel." Both are exact. The proofs are based on a classical special case of this formula. The methods are similar to those used in two earlier papers by Besicovitch and by the author. See Landau's Cambridge Tract and the author's *Bemerkungen zu einer Note von Besicovitch*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 185–192. (Received January 19, 1940.)

189. F. K. Schmidt and Saunders MacLane: *On inseparable fields.*

The original proof of the structure theorem for p-adic fields depended on an analysis of the residue class field by means of certain Steinitz field towers. Saunders Mac Lane has given examples to show that the original analysis of these towers could not be correct (Transactions of this Society, vol. 46 (1939), pp. 23–45). The authors now develop an alternative treatment of modified Steinitz towers, in terms of which the proof of the structure theorem can be carried out. The first modification is the device of replacing a denumerable transcendence basis $t_0, t_1, t_2, \cdots$ by another basis $t_0, t_1^p, t_2^{p^2}, \cdots$, which is almost a p^mth power. The second modification consists in generating an arbitrary extension K preserving p-independence over a modular field

k by a transfinite sequence of extensions, each of which has a denumerable transcendence basis and preserves p-independence. (Received January 29, 1940.)

190. A. R. Schweitzer: *Concerning general abstract relational spaces.*

The author constructs a set S of postulates for relational spaces in terms of an undefined transitive and symmetrical relation K_H effective between $(n+1)$-ads of elements ($n=1, 2, 3, \cdots$) and invariant under any arbitrarily selected subgroup H of the symmetric group G on $n+1$ elements, including G. When H is the alternating group and $n=3$ the set S is equivalent to Axioms 1–6, 11–13, 16 of the author's descriptive system ${}^3K_3$ (American Journal of Mathematics, vol. 31 (1909), pp. 395–396). A feature of the general theory is the generalized axiom of dimensionality (generalizing Axiom 16, above) which expresses disjunctively relative classification of $(n+1)$-ads corresponding to the classification of the substitutions of the symmetric group G on $n+1$ elements into the subgroup H and cosets with respect to H. Particular instances of the preceding relational spaces are rotation spaces of the regular polygons and polyhedra and other metric spaces. When H coincides with G, instances include the author's system of postulates ${}^nK^*$ for projective geometry (this Bulletin, abstract 42-11-443). (Received January 28, 1940.)

191. W. T. Scott: *Approximation to real irrationals by certain classes of rational fractions.*

The set of all irreducible rational fractions is divided into three classes, according as the numerator and denominator are even or odd integers. For each of these classes it is found that there are infinitely many fractions p/q satisfying $|\omega - p/q| < k/q^2$ when $k \geq 1$, regardless of the value of the real irrational number ω. If $k<1$, there exist irrationals everywhere dense on the real axis for which the inequality holds for at most a finite number of fractions of a given one of the three classes. The method of proof depends on the geometric properties of elliptic modular transformations. (Received January 6, 1940.)

192. Seymour Sherman: *A comparison of linear measures of point sets in the plane.* Preliminary report.

The purpose of this investigation is to compare the linear measures defined by Carathéodory and Deltheil. H. Steinhaus (Comptes Rendus du Premier Congrès des Mathématiciens des Pays Slaves, 1929, pp. 348–354) has proved that the Deltheil measure of a rectifiable plane Jordan curve is twice its length and so twice its Carathéodory linear measure. Here it is shown that, contrary to Steinhaus's belief, there exist sets (in particular every set irregular in the sense of Besicovitch) with Deltheil measure zero and Carathéodory measure nonzero. In general if a set has finite Carathéodory linear measure, then it is Deltheil measurable with Deltheil measure equal to twice the Carathéodory linear measure of its regular part. Deltheil measurability does not imply linear measurability in the sense of Carathéodory. Nothing is proved concerning sets of noncountably infinite Carathéodory linear measure and no extensions are made to sets in more general product spaces or even to sets in higher dimensional euclidean spaces. (Received January 11, 1940.)

193. M. F. Smiley: *Measurability and modularity in the theory of lattices.*

In a previous note (*A note on measure functions in a lattice*, to appear in this Bulletin), a notion of measurability (with respect to a function μ) of elements of an arbi-

trary lattice was introduced. Results of V. Glivenko (*Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 933–934) and of L. R. Wilcox and the author (*Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 313) indicate that the idea of measurability and that of modularity are intimately related. The purpose of this note is to exhibit a further relationship which does not depend on metric properties of the function μ. Under the assumption that $a \leqq b$ with $\mu(a) = \mu(b)$ implies that $a = b$, it is found that the measurability of each subelement of a fixed element c implies the complete modularity of each subelement of c. A further consequence is the proof of a theorem in the reverse direction in case the lattice is of finite dimension and satisfies the conclusion of Jordan's theorem. (Received February 1, 1940.)

194. Gabor Szegö: *Remarks on a note of R. Wilson and related subjects.*

Let $w(x)$ be a weight function in $[-1, +1]$ and $\{p_n(x) = k_n x^n + \cdots\}$ the associated set of orthonormal polynomials. (1) Assuming the existence of $\int_0^\pi \log w(\cos\theta)d\theta$, a simple proof is given for the relations $\lim_{n\to\infty} \max |p_n(x)|^{1/n} = (1/2) \cdot \lim_{n\to\infty} k_n^{1/n} = 1$, $-1 \leqq x \leqq +1$, recently discussed by R. Wilson (this Bulletin, vol. 45 (1939), p. 190). (2) An older theorem of Shohat stating that the equivalency of the relations $2^{-n}k_n = O(1)$, $\lim_{n\to\infty} 2^{-n}k_n$ exists is proved in an arrangement slightly different from that of Shohat. (3) It is shown that the relation $2^{-n}k_n = O(1)$ (or the existence of $\lim_{n\to\infty} 2^{-n}k_n$) is equivalent to the existence of the integral mentioned in (1). (Received January 22, 1940.)

195. G. B. Thomas: *Regular positive ternary quadratic forms.* Preliminary report.

Let $f = f(x, y, z) = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$ be a positive, reduced, properly or improperly primitive form with integral coefficients and Hessian $H = \Omega^2\Delta$, where Ω is the g.c.d. of the 2-rowed minors of H. Then f is said to be *regular* if the integers not represented by f coincide with the positive integers in certain sets of arithmetic progressions. (L. E. Dickson, Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341.) The problem of determining all such regular forms with $r = s = t = 0$ has been solved. (Dickson, loc. cit.; B. W. Jones and Gordon Pall, Acta Mathematica, vol. 70 (1939), pp. 165–191.) Combining some results due to B. W. Jones (Transactions of this Society, vol. 33 (1931), pp. 92–124) with certain theorems regarding primes in arithmetic progressions, the present paper shows that there are not more than 99 nor fewer than 94 regular forms with $\Omega = 1$ and r, s, t not all zero. The regular forms are listed, together with the progressions of integers they do not represent. (Received January 16, 1940.)

196. Leonard Tornheim: *Linear forms in function fields.*

The following analogue for function fields of a well known theorem of Minkowski on linear forms is proved algebraically. Let L_i be n linear expressions in n variables x_j with coefficients a_{ij} in $F(z)$ where z is an indeterminate over a field F and with determinant $|a_{ij}|$ of degree d. If the sum of n integers c_i is greater than $d - n$, there exists a set of values for the x_j in $F[z]$ such that each L_i has degree at most c_i. (Received January 15, 1940.)

197. H. S. Vandiver: *Elements of a theory of abstract discrete semigroups.*

A set of axioms is given for a semi-group, the latter being a system elsewhere roughly described by the author as a set closed under an associative operation. A

feature of the treatment employed is the fact that the parentheses are not used, there being an axiom of substitution which leads to a result which is equivalent to the associated law, but which does not contain parentheses in its statement. The symbol of relation ($\cong$) (called equivalence) does not necessarily stand for equality in the usual sense; for example, the integers 1, 2, 3, 4 are said to form a semi-group under multiplication, modulo 5. In this case equivalence means "is congruent to, modulo 5." The proofs of a number of theorems given without proof in a former article (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 552–555) are included in this paper. A theorem is also proved concerning semi-groups, which includes Dedekind's well known theorem of inversion. (Received December 30, 1939.)

198. H. S. Vandiver: *On general methods for obtaining congruences involving Bernoulli numbers.*

This paper has been published (this Bulletin, vol. 46 (1940), pp. 121–123).

199. Abraham Wald: *Asymptotically most powerful tests of statistical hypotheses.*

Let $F(x, \theta)$ be the distribution of a variate X involving an unknown parameter θ. For testing the hypothesis $\theta=\theta_0$, it is necessary to choose a region of rejection W_n in the n-dimensional sample space. Denote by $P(W_n|\theta)$ the probability that the sample point will fall in W_n under the assumption that θ is the true value of the parameter. For any region U_n denote by $g(U_n)$ the greatest lower bound of $P(U_n|\theta)$. For any pair of regions U_n and T_n denote by $L(U_n, T_n)$ the least upper bound of $P(U_n|\theta)-P(T_n|\theta)$. An infinite sequence $\{W_n\}$ of regions is said to be an asymptotically most powerful test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$ and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$, lim sup $L(Z_n, W_n)\leqq 0$ holds. A sequence of regions $\{W_n\}$ is said to be an asymptotically most powerful unbiased test of the hypothesis $\theta=\theta_0$ on the level of significance α if $P(W_n|\theta_0)=\alpha$, lim inf $g(W_n)\geqq\alpha$, and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n|\theta_0)=\alpha$ and lim inf $g(Z_n)\geqq\alpha$ the inequality lim sup $L(Z_n, W_n)\leqq 0$ holds. It has been shown that the commonly used tests, based on the maximum likelihood estimate, are asymptotically most powerful. (Received February 5, 1940.)

200. William Wernick: *Complete sets of logical functions.*

H. M. Sheffer has shown that the 16 functions of the calculus of sentences can all be defined by the single function "stroke" which may have a dual interpretation. Other authors have shown that certain pairs of undefined functions can define this calculus, and that certain other pairs cannot. The set of all functions which can be defined in terms of members of a given undefined set of functions is called the field of that given set. Fields of all single functions and of pairs, triads, and tetrads of functions are discussed, using known properties of symmetry, duality, and transitivity. A set of undefined functions is called *"complete"* if it is *sufficient* (all the 16 functions can be defined in terms of members of the set) and *non-redundant* (no member of the set can be defined in terms of other members of the set). The existence of 2 complete sets of single functions, 24 complete pairs, and 6 complete triads is shown. It is proved that there exists no complete set of four or more functions, by showing that if such a set is sufficient it must be redundant. (Received January 24, 1940.)

201. Hermann Weyl: *Theory of reduction for arithmetical equivalence.*

In the frame of Minkowski's geometry of numbers, reduction appears as the

problem to normalize a basis for the n-dimensional lattice of vectors with integral components in terms of a given convex body. The theory in this generality can be carried through much further than Minkowski seems to have assumed, by means of his geometric methods. He limited himself to the special case of ellipsoids (quadratic forms), at the same time abandoning his own geometric approach. Even for this special case the new treatment induces considerable simplification, closes certain gaps, and results in sharper estimates. The theory is carried over to vectors whose components are complex numbers or quaternions, with an integrity domain of class number 1 serving for the definition of lattice vectors. The quadratic forms are then replaced by Hermitian and "Hamiltonian" forms respectively. (Received January 24, 1940.)

202. John Williamson: *An algebraic problem involving the involutory integrals of linear dynamical systems.*

It is known that a conservative Hamiltonian system with n degrees of freedom has exactly n independent conservative integrals which form an involutory system. It is here shown that, if the system is linear, these n integrals may be chosen as quadratic forms. It is also shown that, if $2H$ is the symmetric matrix, which is the Hessian of the Hamiltonian function, and G the matrix of the covariant bilinear form, that is, the standard canonical form of a nonsingular skew symmetric matrix, the total number of independent conservative quadratic integrals is $2n-m$, where either $2m$ or $2m+1$ is the order of the minimal equation of HG^{-1}. In case HG^{-1} is nonderogatory, the n independent quadratic integrals may be so chosen that their matrices are the symmetric matrices $(HG^{-1})^kH$, $(k=0, 1, 2, \cdots, n-1)$. These n integrals in this case form a set in involution. (Received January 30, 1940.)

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